454 Midterm 2 Solutions[1]

1. Question 1

TRUE/FALSE

(a) Let $K\subseteq \mathbf{R}^2$ be a convex set. Let $f\colon K\to K$ be continuous. Then f has a fixed point. That is, there exists $k\in K$ such that $f(k)=k$.

FALSE. Let $K=\mathbf{R}^2$. Then K is convex. Consider $f(x,y)=(x+1,y)$ for all $(x,y)\in\mathbf{R}^2$. Then f is continuous (it is an affine function), but it has no fixed point, since if $f(x,y)=(x+1,y)=(x,y)$ for some $(x,y)\in\mathbf{R}^2$, then $x+1=x$ i.e. $1=0$, a contradiction.

(b) Let $K=\{(x,y)\in\mathbf{R}^2\colon x^2+y^2=1\}$. Then K is convex.

FALSE. $(1,0)$ and $(-1,0)$ are in K, but $(1/2)((1,0)+(-1,0))=(0,0)\notin K$.

(c) Let $A,B\subseteq\mathbf{R}^2$ be convex sets. Then $A\cup B$ is convex.

FALSE. Let A be the x axis and let B be the y axis. (So $A=\{(x,y)\in\mathbf{R}^2\colon y=0\}$, $B=\{(x,y)\in\mathbf{R}^2\colon x=0\}$.) Then A,B are convex, but $A\cup B$ is not, since $(1,0)\in A$, $(0,1)\in B$, but $(1/2)((1,0)+(0,1))=(1/2,1/2)$ is in neither A nor B.

[There are many other good examples, e.g. take A,B to be two different singleton elements, or let A,B be basically any two disjoint convex sets.]

(d) Any general sum game has at least two Nash equilibria.

FALSE. We showed in the Prisoner's Dilemma that there is only one Nash equilibrium.

2. Question 2

In the simplified game of poker, we computed the expected payoffs of the players to be the following.

		Player II: Call	Player II: Fold
Player I	BB	$(0,0)$	$(1,-1)$
	FB	$(1/2,-1/2)$	$(0,0)$
	BF	$(-3/2,3/2)$	$(0,0)$
	FF	$(-1,1)$	$(-1,1)$

(Each entry of this matrix is of the form (a,b) where a is the payoff for player I and b is the payoff for player II.)

- Prove that a Nash equilibrium exists for these payoffs.
- Using domination, show that two of the rows of this payoff matrix can be ignored (i.e. two of Player I's strategies are played with probability 0).
- Prove that there exists a Nash equilibrium that is not pure. (You do not have to write down what this Nash equilibrium is, just show it is not pure.)

Solution. The third and fourth rows are dominated by the first row for Player I, since $(0,1)\geq(-3/2,0)$ and $(0,1)\geq(-1,-1)$. So, the third and fourth rows can be ignored, i.e. we have reduced to the payoff matrix

By Nash's Theorem, a Nash equilibrium exists for this smaller payoff matrix, and it corresponds to strategies in the original game where the third and fourth rows are played with probability zero each.

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Player I		Player II: Call	Player II: Fold
	BB	$(0,0)$	$(1,-1)$
	FB	$(1/2,-1/2)$	$(0,0)$

We can see by inspection that the Nash equilibrium is not pure.

- The top left corner is not a Nash equilibrium, since Player I would switch from BB to FB.
- The top right corner is not a Nash equilibrium, since Player II would switch to Call instead of Fold.
- The bottom left corner is not a Nash equilibrium, since Player II would switch to Fold instead of Call.
- The bottom right corner is not a Nash equilibrium, since Player I would switch to BB instead of FB.

Since there are no pure Nash equilibria, there must be at least one mixed Nash equilibria, by Nash's Theorem.

3. Question 3

Define $v\colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of v. JUSTIFY YOUR ANSWER.

Solution. We first claim that $\phi_1(v) = \phi_2(v) = \phi_3(v)$. This will follow by an application of Axiom (i). We need to check the assumption of Axiom (i) holds for all eight subsets S of $\{1,2,3\}$. When $S = \emptyset$, it is given that $v(\{1\}) = v(\{2\}) = v(\{3\})$. When $S = \{1\}$, we know $v(\{1,2\}) = v(\{1,3\})$; similarly the assumption of Axiom (i) holds for $S = \{2\}$ and $S = \{3\}$. The remaining assumptions of Axiom (i) hold vacuously (in the case that $|S| \geq 2$). We conclude that Axiom (i) tells us $\phi_1(v) = \phi_2(v) = \phi_3(v)$. Now, from Axiom (iii), $\phi_1(v) + \phi_2(v) + \phi_3(v) = v(\{1,2,3\}) = 1$. In conclusion, $\phi_1(v) = \phi_2(v) = \phi_3(v) = 1/3$.

4. Question 4

Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0,1]$ in a sealed-bid second price auction. That is, V_1 and V_2 are uniformly distributed in the interval $[0,1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\, v \in [0,1]$. That is, each player will bid exactly their private value.

Solution 1. Suppose player 2 bids $\beta_2(v) = v$ and has private valuation V_2. We consider whether or not player 1 can gain by deviating from the equilibrium strategy. Suppose player 1 has valuation v and bids b. Since we're in a second price auction, the winner pays the bid of their opponent. Note player 1 wins the auction if $b > V_2$ and loses if $b < V_2$. Thus the payoff for player 1 is

$$\begin{cases} 0 & \text{if } b < V_2, \\ v - V_2 & \text{if } b \geq V_2. \end{cases}$$

Now if $v < V_2$, player 1 wants to lose the auction, since the payoff if they won would be negative (moreover bidding $b = v$ will cause player 1 to lose the auction). However, if $v > V_2$, player 1 wants to win the auction, and receives the same payoff regardless of the bid b, as long as the bid exceeds V_2 (moreover bidding $b = v$ will cause player 1 to win the auction). The choice of b that always accomplishes these objectives is $b = v$, so $\beta_1(v) = v$.

By symmetry, if we assume $\beta_1(v) = v$, then we can conclude as above that $\beta_2(v) = v$. That is, bidding their private value is a Bayes-Nash equilibrium.

Solution 2. Suppose each bidder bids their private value. Then each buyer with this strategy maximizes their profit, *regardless of what the other players do.* This follows since a buyer with private value v can make a profit at most $\max(v-m, 0)$ where m is the maximum of all other bids, and this profit is achieved when a buyer bids their private value. Note that this conclusion does not require any assumptions on the buyers, such as independence of private values.

In particular, suppose player 2 bids $\beta_2(v) = v$ and has private valuation V_2. We consider whether or not player 1 can gain by deviating from the equilibrium strategy. From the previous argument, player 1 maximizes their profit by also bidding $\beta_1(v) = v$. It follows that bidding their private value is a Bayes-Nash equilibrium.