Math 454, Spring 2026, USC — Instructor: Steven Heilman

Name: ____________ USC ID: ____________ Date: ____________

Signature: ____________. Discussion Section: ____________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	8	
2	10	
3	10	
4	10	
Total:	38	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let m, n be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let A be the payoff matrix for player I and let B be the payoff matrix for player II. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m, \qquad \widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

If (x, y) is a Nash equilibrium, and if x has one coordinate equal to 1 with all other coordinates 0, and if y has one coordinate equal to one with all other coordinates zero, then the equilibrium (x, y) is called **pure**. If (x, y) is a Nash equilibrium that is not pure, then it is called **mixed**.

A set $K \subseteq \mathbf{R}^n$ is called **convex** if, for any $x, y \in K$ and for any $0 \leq t \leq 1$, we have $tx + (1-t)y \in K$. A set $K \subseteq \mathbf{R}^n$ is called **bounded** if there exists $r > 0$ such that $||x|| \leq r$ for all $x \in K$.

Suppose we have a game with n players together with a characteristic function $v \colon 2^{\{1,\ldots,n\}} \to \mathbf{R}$. For each $i \in \{1, \ldots, n\}$, we define the **Shapley value** $\phi_i(v) \in \mathbf{R}$ to be any set of real numbers satisfying the following four axioms:

(i) (Symmetry) If for some $i, j \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq \{1, \ldots, n\}$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.

(ii) (No power/ no value) If for some $i \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \{1, \ldots, n\}$, then $\phi_i(v) = 0$.

(iii) (Additivity) If u is any other characteristic function, then $\phi_i(v + u) = \phi_i(v) + \phi_i(u)$, for all $i \in \{1, \ldots, n\}$.

(iv) (Efficiency) $\sum_{i=1}^{n} \phi_i(v) = v(\{1, \ldots, n\})$.

We define a **symmetric auction**. A single object is for sale at an auction. The seller is willing to sell the object at any nonnegative price. There are n buyers, which we identify with the set $\{1, 2, \ldots, n\}$. All buyers have some set of **private values** in $[0, \infty)$. We denote the private value of buyer $i \in \{1, \ldots, n\}$ by V_i, so that V_i is a random variable that takes nonnegative real values. We assume that all of the random variables $V_1, \ldots, V_n$ are independent. We also assume that $V_1, \ldots, V_n$ are identically distributed, with a continuous density function. That is, there exists some continuous function $f\colon \mathbf{R} \to [0,\infty)$ with $\int_{-\infty}^{\infty} f(x)dx = 1$ such that: for each $i \in \{1,\ldots,n\}$, for each $t \in \mathbf{R}$, the probability that $V_i \leq t$ is equal to $\int_{-\infty}^{t} f(x)dx$. We also assume that all buyers are **risk-neutral**, so that each buyer seeks to maximize their expected profits.

Finally, we assume that all of the above assumptions are **common knowledge**. That is, every player knows the above assumptions; every player knows that every player knows the above assumptions; every player knows that every player knows that every player knows the above assumptions; etc.

Under the above assumptions, a **pure strategy** for Player $i \in \{1,\ldots,n\}$ is a function $\beta_i\colon [0,1] \to [0,\infty)$. So, if Player i has a private value of V_i, he will make a bid of $\beta_i(V_i)$ in the auction. (We will not discuss mixed strategies in auctions.)

Given the strategies $\beta = (\beta_1,\ldots,\beta_n)$, and given any $v \in [0,1]$, Player i has expected profit $P_i(\beta, v)$, if her private value is v. (If buyer i wins the auction, and if buyer i has private value v and bid b, then the profit of buyer i is $v - b$.) We say that a strategy β is an **equilibrium** if, given any $v \in [0,1]$, any $b \geq 0$, and any $i \in \{1,\ldots,n\}$,

$$P_i(\beta, v) \geq P_i((\beta_1,\ldots,\beta_{i-1}, b, \beta_{i+1},\ldots,\beta_n), v).$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **EXPLAIN YOUR REASONING**. If the statement is false, **PROVIDE A COUNTEREXAMPLE AND EXPLAIN YOUR REASONING**. Unlike other questions on this exam, you may cite homework problems to complete this part of the exam.

   [most of these were modified homework problems]

   (a) (2 points) Let $K \subseteq \mathbf{R}^2$ be a convex set. Let $f\colon K \to K$ be continuous. Then f has a fixed point. That is, there exists $k \in K$ such that $f(k) = k$.

   TRUE FALSE (circle one)

   (b) (2 points) Let $K = \{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 = 1\}$. Then K is convex.

   TRUE FALSE (circle one)

   (c) (2 points) Let $A, B \subseteq \mathbf{R}^2$ be convex sets. Then $A \cup B$ is convex.

   TRUE FALSE (circle one)

   (d) (2 points) Any general sum game has at least two Nash equilibria.

   TRUE FALSE (circle one)

2. (10 points) In the simplified game of poker, we computed the expected payoffs of the players to be the following.

		Player II	
		Call	Fold
Player I	BB	$(0,0)$	$(1,-1)$
	FB	$(1/2,-1/2)$	$(0,0)$
	BF	$(-3/2,3/2)$	$(0,0)$
	FF	$(-1,1)$	$(-1,1)$

(Each entry of this matrix is of the form (a, b) where a is the payoff for player I and b is the payoff for player II.)

- Prove that a Nash equilibrium exists for these payoffs.
- Using domination, show that two of the rows of this payoff matrix can be ignored (i.e. two of Player I's strategies are played with probability 0).
- Prove that there exists a Nash equilibrium that is not pure. (You do not have to write down what this Nash equilibrium is, just show it is not pure.)

[this was a modified example from class]

3. (10 points) Define $v\colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

   Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of v. JUSTIFY YOUR ANSWER.

   [this was a repeated practice exam questions]

4. (10 points) Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid second price auction. That is, V_1 and V_2 are uniformly distributed in the interval $[0, 1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\ v \in [0, 1]$. That is, each player will bid exactly their private value.

   [this was an example we did in class]

(Scratch paper)