454 Midterm 1 Solutions[1]

1. Question 1

TRUE/FALSE

(a) In the game of chess, it is known that the first player has a winning strategy. That is, the first player can guarantee a win, regardless of what the second player does.

FALSE. Zermelo's Theorem implies that one of three alternatives holds, but it is not known which holds. (The first player has a winning strategy, or the second player does, or both players have a strategy guaranteeing at least a draw.) (Recall that chess is a progressively bounded, partisan game.)

(b) Suppose the game of Nim begins with one pile of 9999 chips and one pile of 10000 chips. Then the first player has a winning strategy.

TRUE. The first move is to take one chip from the pile of 10000, resulting in the game position $(9999, 9999)$. No matter what the other player does, the first player can then force both piles to have an equal number of chips, resulting in a win for the first player (since eventually the first player encounters a single pile of chips).

(c) Let A be a real 10×10 matrix. Then

$$\max_{x\in\Delta_{10}} \min_{y\in\Delta_{10}} x^T A y = \min_{y\in\Delta_{10}} \max_{x\in\Delta_{10}} x^T A y.$$

TRUE. This follows from Von Neumann's Minimax Theorem.

(d) Every two-player zero-sum game has an optimal strategy.

TRUE. This follows from the extreme value theorem. The definition of optimal strategy involves maximizing or minimizing a continuous function on a closed and bounded set, and the respective max or min is guaranteed to exist by the extreme value theorem.

2. Question 2

Prove the following. On a standard Hex game board, the first player has a winning strategy. That is, the first player has a strategy that guarantees a win, regardless of what the second player does.

Solution. The game cannot end in a tie, and Hex on a finite game board is progressively bounded (since each player fills a hexagon in each turn, the game ends in a number of moves less than or equal to the number of empty cells in the starting game board.) So by Zermelo's Theorem, one player has a (memoryless) winning strategy. We argue by contradiction. Suppose the second player has a winning strategy. Note that the game positions are symmetric with respect to swapping the colors of the game board. So, the first player can use the second player's winning strategy as follows. On the first move, the first player just colors in any hexagon H. On the first player's next move, they just use the winning strategy of the second player (pretending that hexagon H is not filled in). If the first player is ever required to color the hexagon H, they just color any other hexagon instead, if an unfilled hexagon exists. Having this extra hexagon filled can only benefit the first player. (This extra filled in hexagon H' can now function like the previous H hexagon, i.e. Player I can pretend H' is empty in order to use Player II's strategy, until this strategy tells player I to fill in H', then player I fills in another empty hexagon H", and so on.) So, the first player is guaranteed to win, and both players will win, a contradiction to what we showed in class or the notes. We conclude the first player has a winning strategy.

3. Question 3

Let $K := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon (x_1 - 4)^2 + (x_2 - 4)^2 + (x_3 - 4)^2 \leq 1\}$.

- Show that $(0, 0, 0) \notin K$.
- Prove that there exists some $z \in \mathbf{R}^3$ and $c > 0$ such that

  $$z^T x > c, \qquad \forall\, x \in K.$$

  (You may assume that K is closed and convex without proof, i.e. you do not have to prove that K is closed and convex.)
- Write down an explicit form of z such that

  $$z^T x > 1, \qquad \forall\, x \in K.$$

  Justify your answer. (For example, maybe $z = (-5, 3, 1)$ works.)

Solution. Note that $(0-4)^2 + (0-4)^2 + (0-4)^2 = 3 \cdot 16 = 48 > 1$, so $(0, 0, 0) \notin K$. Since $0 \notin K$ and K is closed and convex, the existence of z is guaranteed by the Separating Hyperplane Theorem. Moreover, we can check e.g. that $z = (1, 1, 1)$ is sufficient. This follows since any $x \in K$ necessarily satisfies $|x_i - 4| \leq 1$ for each $1 \leq i \leq 3$ (otherwise the inequality defining K is violated), i.e. $3 \leq x_i \leq 5$ for all $x \in K$, for all $1 \leq i \leq 3$, which means that

$$(1, 1, 1)^T x = x_1 + x_2 + x_3 \geq 3 + 3 + 3 = 9 > 1.$$

That is $z^T x > 1\ \forall\, x \in K$..

4. Question 4

Let $n \geq 2$ be an integer. Prove that Δ_n is convex and bounded.

Solution. Let $x, y \in \Delta_n$ and let $t \in [0, 1]$. By definition of Δ_n, $\sum_{i=1}^n x_i = 1$ and $x_i \geq 0$ for all $1 \leq i \leq n$, and $\sum_{i=1}^n y_i = 1$ and $y_i \geq 0$ for all $1 \leq i \leq n$. Since $t + (1 - t) = 1$, we then have

$$\sum_{i=1}^n t x_i + \sum_{i=1}^n (1 - t) y_i = t(1) + (1 - t)1 = t + (1 - t) = 1.$$

That is, $tx + (1 - t)y$ satisfies $\sum_{i=1}^n (tx + (1 - t)y)_i = 1$. Since $t, 1 - t \geq 0$, and $x_i, y_i \geq 0$ for all $1 \leq i \leq n$, we have $t x_i + (1 - t) y_i \geq 0$ for all $1 \leq i \leq n$. Then is $(tx + (1 - t)y)_i \geq 0$ for all $1 \leq i \leq n$. In summary, we have shown that $tx + (1 - t)y \in \Delta_n$. That is, Δ_n is convex.

For boundedness, we use $0 \leq x_i \leq 1$, so that $x_i^2 \leq 1$ for all $1 \leq i \leq n$ to get

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n 1} = \sqrt{n}.$$

So, we may choose $r := \sqrt{n}$ to see that Δ_n is bounded. We could also use $r = 1$, since $0 \leq x_i \leq 1$ implies $0 \leq x_i^2 \leq x_i$ for all $1 \leq i \leq n$, so

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n x_i} = \sqrt{1} = 1,$$

using $x \in \Delta_n$ in the penultimate equality.

5. Question 5

Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 1 & 4 & 7 & 1 \\ 2 & 5 & 9 & 10 \\ 3 & 6 & 9 & 0 \end{pmatrix}$$

Describe optimal strategies for this game.

Solution. The second row dominates the first row, so we can ignore the first row for the purpose of computing the value of the game. That is, we can equivalently compute the value of the matrix

$$\begin{pmatrix} 2 & 5 & 9 & 10 \\ 3 & 6 & 9 & 0 \end{pmatrix}$$

The first column is dominated by the second and third columns, so the second and third columns can be ignored. That is, we have reduced to computing the value of the matrix

$$A := \begin{pmatrix} 2 & 10 \\ 3 & 0 \end{pmatrix}.$$

We then compute

$$\max_{x \in \Delta_2} \min_{y \in \Delta_2} x^T A y = \max_{s \in [0,1]} \min_{t \in [0,1]} (s, 1-s) A \begin{pmatrix} t \\ 1-t \end{pmatrix} = \max_{s \in [0,1]} \min_{t \in [0,1]} [2st + 10s(1-t) + 3(1-s)t]$$
$$= \max_{s \in [0,1]} \min_{t \in [0,1]} [10s + 3t - 11st] = \max_{s \in [0,1]} \min(10s, -s+3)$$

(In the last equality, we use that the function $t \mapsto 10s + 3t - 11st$ is linear in t, so the minimum over $t \in [0,1]$ must occur when $t = 0$ or $t = 1$.) The maximum of the function $s \mapsto \min(10s, -s+3)$ occurs when $10s = -s + 3$, i.e. when $11s = 3$, i.e. $s = 3/11$. That is, the optimal strategy for Player I for the matrix A is $(3/11, 8/11)$. Similarly,

$$\min_{y \in \Delta_2} \max_{x \in \Delta_2} x^T A y = \min_{t \in [0,1]} \max_{s \in [0,1]} [10s + 3t - 11st] = \max_{s \in [0,1]} \max(3t, 10 - 8t)$$

The minimum of the function $t \mapsto \max(3t, 10-8t)$ occurs when $3t = 10 - 8t$, i.e. when $11t = 10$, i.e. $t = 10/11$. That is, the optimal strategy for Player II for the matrix A is $(10/11, 1/11)$.

Returning to the original matrix, these strategies for Players I and II correspond to the second and third rows, and the first and last columns. So, a pair of optimal strategies for this game is:

$$x = (0, 3/11, 8/11), \qquad y = (10/11, 0, 0, 1/11).$$

Finally, the value of the game can be found in various ways. We could e.g. use that it is equal to the minmax value of the matrix A, which gives a value of $30/11$ (after plugging in $t = 10/11$) into the function $\max(3t, 10 - 8t)$.