Math 454, Spring 2026, USC                    Instructor: Steven Heilman

Name: _____     USC ID: _____     Date: _____

Signature: _____. Discussion Section: _____
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

| Problem | Points | Score |
|---------|--------|-------|
| 1 | 8 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 48 | |

Do not write in the table to the right. Good luck![a]

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**Reference sheet**

Below are some definitions that may be relevant.

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An **impartial combinatorial game** is a combinatorial game with two players who both have the same set of legal moves. A **terminal position** is a position from which there are no legal moves. Every non-terminal position has at least one legal move. Under **normal play**, the player who moves to the terminal position wins.

For any impartial combinatorial game, let **N** (for "next") be the set of game positions such that the first player to move can guarantee a win. Let **P** denote the set of game positions such that *any* legal move leads to a position in **N**. We also let **P** contain all terminal positions.

Let $m$ be a positive integer.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^m x_i = 1, \ x_i \geq 0, \ \forall\, 1 \leq i \leq m\}.$$

Let $m, n$ be positive integers. Let $A$ be an $m \times n$ real matrix. Then the **value** of the two-person zero-sum game with payoff matrix $A$ is

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y.$$

A mixed strategy $\widetilde{x} \in \Delta_m$ is **optimal for player** $I$ if

$$\min_{y \in \Delta_n} \widetilde{x}^T A y = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y.$$

A mixed strategy $\widetilde{y} \in \Delta_n$ is **optimal for player** $II$ if

$$\max_{x \in \Delta_m} x^T A \widetilde{y} = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y.$$

We say the pair $(\widetilde{x}, \widetilde{y})$ are **optimal strategies** for the payoff matrix $A$ if $\widetilde{x} \in \Delta_m$ is optimal for player $I$ and $\widetilde{y} \in \Delta_n$ is optimal for player $II$.

For any $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$, we denote

$$||x|| := \Big( \sum_{i=1}^m x_i^2 \Big)^{1/2}.$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) In the game of chess, it is known that the first player has a winning strategy. That is, the first player can guarantee a win, regardless of what the second player does.

   <div align="center">TRUE     FALSE    (circle one)</div>

   (b) (2 points) Suppose the game of Nim begins with one pile of 9999 chips and one pile of 10000 chips. Then the first player has a winning strategy.

   <div align="center">TRUE     FALSE    (circle one)</div>

   (c) (2 points) Let $A$ be a real $10 \times 10$ matrix. Then

   $$\max_{x \in \Delta_{10}} \min_{y \in \Delta_{10}} x^T A y = \min_{y \in \Delta_{10}} \max_{x \in \Delta_{10}} x^T A y.$$

   <div align="center">TRUE     FALSE    (circle one)</div>

   (d) (2 points) Every two-player zero-sum game has an optimal strategy.

   <div align="center">TRUE     FALSE    (circle one)</div>

2. (10 points) Prove the following. On a standard Hex game board, the first player has a winning strategy. That is, the first player has a strategy that guarantees a win, regardless of what the second player does.

[This was from the notes/class]

3. (10 points) Let $K := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1 - 4)^2 + (x_2 - 4)^2 + (x_3 - 4)^2 \leq 1\}$.

- Show that $(0, 0, 0) \notin K$.
- Prove that there exists some $z \in \mathbf{R}^3$ and $c > 0$ such that

$$z^T x > c, \qquad \forall\, x \in K.$$

  (You may assume that $K$ is closed and convex without proof, i.e. you do not have to prove that $K$ is closed and convex.)

- Write down an explicit form of $z$ such that

$$z^T x > 1, \qquad \forall\, x \in K.$$

  Justify your answer. (For example, maybe $z = (-5, 3, 1)$ works.)

[This was a modification of class material.]

4. (10 points) Let $n \geq 2$ be an integer. Prove that $\Delta_n$ is convex and bounded.

(Recall a set $K \subseteq \mathbf{R}^n$ is convex if, for any $x, y \in K$ and for any $t \in [0, 1]$, we have $tx + (1 - t)y \in K$.)

(Recall a set $K \subseteq \mathbf{R}^n$ is bounded if there exists $r > 0$ such that $||x|| \leq r$ for all $x \in K$.)

[This was a homework/practice exam question]

5. (10 points) Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 1 & 4 & 7 & 1 \\ 2 & 5 & 9 & 10 \\ 3 & 6 & 9 & 0 \end{pmatrix}$$

Describe optimal strategies for this game.

[This was a modified homework question.]

(Scratch paper)