MATH 425A, ANALYSIS 1, HOMEWORK SOLUTIONS

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Contents

1. Homework 1

Exercise 1.2. Let A, B be subsets of some set X. Define $A^c := \{x \in X : x \notin A\}$. Prove:

$$(A \cap B)^c = A^c \cup B^c.$$

Solution. A possible truth table is as follows:

$x \in A$	$x \in B$	$x \in A \cap B$	$x \in (A \cap B)^c$	$x \in A^c$	$x \in B^c$	$x \in A^c \cup B^c$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Or using various logical operation you can prove the equivalence as follows:

$$\begin{aligned} x \in (A \cap B)^c &\iff \text{not } (x \in A \cap B) \\ &\iff \text{not } (x \in A \text{ and } x \in B) \\ &\iff (\text{not } x \in A) \text{ or } (\text{not } x \in B) \\ &\iff (x \in A^c) \text{ or } (x \in B^c) \\ &\iff x \in (A^c \cup B^c) \end{aligned}$$

□

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Exercise 1.3. Using the Peano axioms, show that the sum of two natural numbers is a natural number.

Solution. We prove that $n + m$ is a natural number using mathematical induction on n.

(Base case) When $n = 0$, we have $0 + m = m \in \mathbb{N}$ for all $m \in \mathbb{N}$ by definition of addition.

(Inductive step) Suppose that $n+m \in \mathbb{N}$ for all $m \in \mathbb{N}$. What we want to prove is that $(n{+}{+}){+}m \in \mathbb{N}$ for all $m \in \mathbb{N}$. But since $n + m \in \mathbb{N}$ by the induction hypothesis, (PA2) shows that

$$(n++) + m = (n + m) ++ \in \mathbb{N}$$

as desired. This proves the inductive step.

Therefore by (PA5), the mathematical induction, it follows that $n+m$ is a natural number for all $n, m \in \mathbb{N}$. □

Exercise 1.4. Using the Peano axioms, show that addition is associative. That is, given natural numbers x, y, z, we have $x + (y + z) = (x + y) + z$. (Hint: fix two of the variables, and induct on the third.) (Note: you can use Lemma 2.9 from the notes.)

Solution. Choose arbitrary $x, y \in \mathbb{N}$. We show that $x + (y + z) = (x + y) + z$ for any $z \in \mathbb{N}$ by inducting on z.

(Base case) When $z = 0$, Lemma 2.7 shows that

$$x + (y + 0) = x + y = (x + y) + 0.$$

This proves the base case.

(Inductive step) Assume that $x + (y + z) = (x + y) + z$ holds for $z \in \mathbb{N}$. Then what we want to prove is that $x + (y + (z++)) = (x + y) + (z++)$. Indeed, utilizing Lemma 2.9 shows that

$$x + (y + (z++)) = x + (y + z)++ = (x + (y + z))++.$$

One the other hand, again by Lemma 2.9, we have

$$(x + y) + (z++) = ((x + y) + z)++.$$

Then the induction hypothesis tells us that they are equal, hence proves the inductive step.

Therefore the claim follows by the mathematical induction. □

Exercise 1.5. Let a, b, c be natural numbers. Using the definition of the order on the natural numbers, prove the following properties.

(1) $a \geq a$.
(2) If $a \geq b$ and $b \geq c$, then $a \geq c$.
(3) If $a \geq b$ and $b \geq a$, then $a = b$.
(4) $a \geq b$ if and only if $a + c \geq b + c$.
(5) $a < b$ if and only if $a + c < b + c$.

Solution.

(a) Since $a = a + 0$ for any $a \in \mathbb{N}$, we have $a \geq a$ by definition.
(b) $a \geq b$ and $b \geq c$ imply that $\exists m, n \in \mathbb{N}$ satisfying $a = b+m$ and $b = c+n$, respectively. Then $a = c + (m + n)$ and hence $a \geq c$.
(c) $a \geq b$ and $b \geq a$ imply that $\exists m, n \in \mathbb{N}$ satisfying $a = b+m$ and $b = a+n$, respectively. Then $a = a + (m + n)$. Applying the cancellation law, we have $m + n = 0$. Then by Corollary 2.18 we obtain $m = n = 0$ and hence $b = a$.

(d) ($\Longrightarrow$): Assume $a \geq b$. Then $a = b + m$ for some $m \in \mathbb{N}$. Adding $c \in \mathbb{N}$ to both sides, we obtain $a + c = (b + c) + m$. This implies $a + c \geq b + c$.

($\Longleftarrow$): Conversely, assume $a + c > b + c$. Then $a + c = (b + c) + m$ for some $m \in \mathbb{N}$. Now appealing to the cancellation law, we obtain $a = b + m$ and thus $a \geq b$ holds.

(e) Constrapositive of the cancellation law shows that $a \neq b$ if and only if $a + c \neq b + c$. Thus

$$a < b \iff a \leq b \text{ and } a \neq b \iff a + c \leq b + c \text{ and } a + c \neq b + c \iff a + c < b + c.$$

□

Exercise 1.6 (The Euclidean Algorithm). Let n be a natural number and let q be a positive natural number. Show that there exist natural numbers m, r such that $0 \leq r < q$ and such that $n = mq + r$. (Hint: fix q and induct on n.)

Solution. Fix $q > 0$. We want to prove that for any $n \in \mathbb{N}$ the following statement holds:

$$\exists m, r \in \mathbb{N} \text{ such that } n = mq + r \text{ and } 0 \leq r < q. \tag{1}$$

To this end we induct on n.

(Base case) $0 = 0 \cdot q + 0$ shows that (1) holds when $n = 0$ with $m = 0$ and $r = 0$.

(Inductive step) Suppose that (1) holds for $n \in \mathbb{N}$. What we want to prove is that the following statement holds: [1]

$$\exists m', r' \in \mathbb{N} \text{ such that } n + 1 = m'q + r' \text{ and } 0 \leq r' < q. \tag{2}$$

Before proving this step, we make an observation that helps us build some intuitions. Using the induction hypothesis (1), we find that

$$n + 1 = (mq + r) + 1 = mq + (r + 1).$$

Comparing this with (2), it is tempting to let $m' = m$ and $r' = r + 1$. It turns out that this fails to work only when $r' \geq q$. This exceptional case happens exactly when $r = q - 1$. (The case $r \geq q$ is automatically excluded by the trichotomy of ordering, together with (1).)

Now let us return to the original proof. With these observations, we divide into two cases:

(Case 1) Suppose that $0 \leq r < q - 1$. We claim that the choice $m' = m$ and $r' = r + 1$ proves (2). Indeed, $0 \leq r' < q$ follows easily. Also, (1) shows that

$$m'q + r' = mq + r + 1 = n + 1.$$

Thus (2) follows with our choice of m' and r'.

(Case 2) Suppose that $r = q - 1$. In this case, we claim that the choice $m' = m + 1$ and $r' = 0$ proves 2. The condition $0 \leq r' < q$ is clear. Also, by (1) we have

$$m'q + r' = (m + 1)q + 0 = mq + q = mq + r + 1 = n + 1.$$

Thus (2) follows with our choice of m' and r' in this case as well.

So we have (2) in both cases and the inductive step follows.

[1] This is just a restatement of (1) with n replaced by $n + 1$. It is introduced to avoid confusion by using different variables.

Therefore by the mathematical induction, (1) holds for any $n \in \mathbb{N}$.

□

Exercise 1.7. Prove the principle of infinite descent. Let $p_0, p_1, p_2, \ldots$ be an infinite sequence of natural numbers such that $p_0 > p_1 > p_2 > \cdots$. Prove that no such sequence exists. (Hint: Assume by contradiction that such a sequence exists. Then prove by induction that for all natural numbers n, N, we have $p_n \geq N$. Use this fact to obtain a contradiction.)

Solution.

We prove this by contradiction. Assume that such a sequence $p_0 > p_1 > p_2 > \ldots$ exists. To derive a contradiction, we claim the following:

Claim. For any $n, N \in \mathbb{N}$ we have $p_n \geq N$.

To this end, we induct on N.

(Base case) Since p_n are natural numbers, we automatically have $p_n \geq 0$ for all $n \in \mathbb{N}$, and the base case follows (by definition of order, $p_n \geq 0$ since $p_n = p_n + 0$.)

(Inductive step) Assume that $p_n \geq N$ for all $n \in \mathbb{N}$. Then by the assumption,

$$p_n > p_{n+1} \geq N$$

and it follows that $p_n \geq N + 1$. This proves the inductive step.

So the claim follows. Then plugging $n = 0$ and $N = p_0 + 1$, it follows that $p_0 \geq p_0 + 1 > p_0$, contradicting the trichotomy of ordering. Therefore no such sequence exists. □

Exercise 1.8. Find a set of integers a_{ij} where $i, j \in \mathbb{N}$ such that $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij}) = 0$, but such that $\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij}) = 1$. (Hint: an example exists where most of the numbers are zero, and the remaining numbers are $+1$ or -1. It may also help to arrange the numbers in a matrix.)

Solution.

Choose (a_{ij}) as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i + 1, \\ 0, & \text{otherwise} \end{cases} \implies (a_{ij}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

On the one hand, for any $i = 1, 2, 3, \ldots$ we have

$$\sum_{j=1}^{\infty} a_{ij} = 0 + \cdots + 0 + 1 + (-1) + 0 + 0 + \cdots = 0.$$

In other words, the sum of i-th row is always 0. Thus we have

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = 0 + 0 + \cdots = 0.$$

On the other hand, for each j it follows that

$$\sum_{i=1}^{\infty} a_{ij} = \begin{cases} 1, & \text{if } j = 1, \\ 0 + \cdots + 0 + (-1) + 1 + 0 + \cdots = 0, & \text{if } j \geq 2. \end{cases}$$

That is, the sum of the first column is 1 and the sum of any other column is 0. Thus we have

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right) = 1 + 0 + 0 + \cdots = 1.$$

Therefore our choice satisfies every requirement. □

2. Homework 2

Exercise 2.1. By breaking into different cases as necessary, prove the following statements. Let x, y be rational numbers. Then $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$. We also have the **triangle inequality**

$$|x + y| \leq |x| + |y|,$$

the bounds

$$-|x| \leq x \leq |x|$$

and the equality

$$|xy| = |x|\,|y|.$$

In particular,

$$|-x| = |x|.$$

Also, the distance $d(x, y) := |x - y|$ satisfies the following properties. Let x, y, z be rational numbers. Then $d(x, y) = 0$ if and only if $x = y$. Also, $d(x, y) = d(y, x)$. Lastly, we have the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

Solution. Before the proof, we observe that $x < y$ implies $-y < -x$. Indeed, this follows by adding $-x - y$ to both sides. Now we prove each statement by splitting into several cases.

- $|x| \geq 0$ for all $x \in \mathbb{Q}$:
  - If $x \geq 0$, then by definition of the absolute value $|x| = x \geq 0$.
  - If $x < 0$, then by definition of the absolute value $0 < -x = |x|$ by definition of the absolute value.

  Thus in any cases we have $|x| \geq 0$.
- $|x| = 0$ if and only if $x = 0$:
  - If $x > 0$, then by definition of the absolute value $|x| = x > 0$ and $x \neq 0$.
  - If $x = 0$, then by definition of the absolute value $|x| = x = 0$ and $x = 0$.
  - If $x < 0$, then by definition of the absolute value $|x| = -x > 0$ and $x \neq 0$.

  Thus we are done.
- $-|x| \leq x \leq |x|$:
  - If $x \geq 0$, then by definition of the absolute value and the definition of order, $|x| = x \geq 0 \geq -x = -|x|$.
  - If $x < 0$, , then by definition of the absolute value and the definition of order, $|x| = -x > 0 > x = -|x|$.

This completes the proof.

- (Triangle inequality) $|x + y| \leq |x| + |y|$:

- If $x+y \geq 0$, then by the previous exercise we have $|x+y| = x+y$. We now claim that $x + y \leq\leq |x| + |y|$, which would complete the proof. To prove this claim, note that $x \leq |x|$ and $y \leq |y|$ from the previous part of the exercise, so $|x|-x \geq 0$ and $|y| - y \geq 0$ by definition of order, i.e. $|x| - x$ and $|y| - y$ are nonnegative rational numbers. So, their sum $|x| - x + |y| - y$ is also a nonnegative rational number. So by definition of order, $|x|+|y|-x-y \geq 0$, i.e. $|x|+|y|-(x+y) \geq 0$, so that $|x| + |y| \geq x + y$ by definition of order.
- If $x+y < 0$, then by the previous exercise we have $|x+y| = -(x+y) = -x-y$. We now claim that $-x - y \leq\leq |x| + |y|$, which would complete the proof. To prove this claim, note that $-x \leq |x|$ and $-y \leq |y|$ from the previous part of the exercise, so $|x| + x \geq 0$ and $|y| + y \geq 0$ by definition of order, i.e. $|x| + x$ and $|y| + y$ are nonnegative rational numbers. So, their sum $|x| + x + |y| + y$ is also a nonnegative rational number, by the definition of the sum of rationals. So by definition of order, $|x| + |y| + x + y \geq 0$, i.e. $|x| + |y| - (-x - y) \geq 0$, so that $|x| + |y| \geq -x - y$ by definition of order.

Thus the claim follows.

- $|xy| = |x||y|$:
  - If $x > 0$ and $y \geq 0$, then x is a positive rational number by definition of order, and y is a nonnegative rational number by definition of order. So the product xy is a nonnegative rational number by the definition of the product of rationals. So by the definition of order again $xy \geq 0$ and $|xy| = xy = |x||y|$. (Since $x > 0$ and $y \geq 0$ we have $x = |x|$ and $y = |y|$ by definition of absolute value)
  - If $x > 0$ and $y < 0$, then $-y > 0$ shows that $-xy = x(-y) \geq 0$ (repeating the justification from the previous part), or equivalently, either $xy < 0$ or $xy = 0$. In any cases, we have $|xy| = -xy = x(-y) = |x||y|$.
  - The case $x < 0$ and $y \geq 0$ is proved in exactly the same way.
  - If $x < 0$ and $y < 0$, then $-x > 0$, $-y > 0$ and thus the first part says $xy = (-x)(-y) > 0$. This yields $|xy| = xy = (-x)(-y) = |x||y|$. (Since $x < 0$ and $y \leq 0$ we have $x = -|x|$ and $y = -|y|$ by definition of absolute value)

  Therefore the claim follows in any cases.
- Since $-1 < 0$, $|-1| = 1$ by definition of absolute value and so from the previous part of the exercise, $|-x| = |(-1)x| = |-1||x| = 1 \cdot |x| = |x|$.

Using the above properties we can prove the metric properties of $d(x, y) := |x - y|$:

- (Non-degeneracy) $d(x, y) = 0$ if and only if $x = y$:

$$d(x, y) = 0 \iff |x - y| = 0 \iff x - y = 0 \iff x = y.$$

  (Recall we showed above that $z \in \mathbb{Q}$ satisfies $|z| = 0$ if and only if $z = 0$, and we used $z = x - y$.)
- (Symmetry) $d(x, y) = d(y, x)$:

$$d(x, y) = |x - y| = |-(x - y)| = |y - x| = d(y, x).$$

  (We showed above that $z \in \mathbb{Q}$ satisfies $|z| = |-z|$, and we used $z = x - y$.)
- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$:

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

(We used the triangle inequality in the form $|a+b| \leq |a| + |b|$ for $a, b \in \mathbb{Q}$ where $a = x - y$ and $b = y - z$.)

This completes the whole proof.

□

Exercise 2.2. Using the usual triangle inequality, prove the **reverse triangle inequality**: For any rational numbers x, y, we have $|x-y| \geq ||x| - |y||$.

Solution. Note that we both have

$$|x| = |(x-y)+y| \leq |x-y| + |y| \quad \text{and} \quad |y| = |(y-x)+x| \leq |y-x| + |x|.$$

Thus it follows that

$$|x| - |y| \leq |x-y| \quad \text{and} \quad |y| - |x| \leq |y-x| = |x-y|.$$

So regardless of either $||x|-|y|| = |x|-|y|$ or $||x|-|y|| = |y|-|x|$, it follows that $||x|-|y|| \leq |x-y|$ as desired. □ □

Exercise 2.3. Let x be a rational number. Prove that there exists a unique integer n such that $n \leq x < n+1$. In particular, there exists an integer N such that $x < N$. (Hint: use the Euclidean Algorithm.)

Solution. Let $x \in \mathbb{Q}$ and write it as a quotient $x = \frac{p}{q}$ of two integers $p, q \in \mathbb{Z}$ with $q > 0$. Then by the Euclidean algorithm, there exists $m, r \in \mathbb{Z}$ with $0 \leq r < q$ such that $p = mq + r$. This gives

$$x = \frac{p}{q} = \frac{mq+r}{q} = m + \frac{r}{q}.$$

But since $0 \leq \frac{r}{q} < 1$, it follows that $m \leq x < m+1$ and this proves the existence.

To prove the uniqueness, let $m, n \in \mathbb{Z}$ satisfy $m < x < m+1$ and $n < x < n+1$, respectively. We claim that in fact $m = n$. By relabeling if required, we may assume that $m \leq n$. Then $n = m + a$ for some $a \in \mathbb{N}$. But if $a \neq 0$, then $a \geq 1$ and this implies

$$x < m+1 \leq m+a = n \leq x \implies x < x,$$

a contradiction! Therefore $a = 0$ and hence $m = n$. □

Exercise 2.4. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals. Prove that $(a_n)_{n=0}^{\infty}$ is bounded.

Solution. The concept of Cauchy sequence is designed to capture the behavior of terms getting closer to each other *ad infinitum*. In particular, (a_n) tends to stabilize as n grows, and it allows to control the size of a tail (a sequence of the form $(a_n)_{n \geq N}$ for some N) with just a single term plus a small error. This motivates us to divide the sequence into two parts, one consisting of leading terms and the other being a tail.

Solution. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence. We want to find a constant $0 < M \in \mathbb{Q}$ that bounds this sequence. Indeed, by the definition of Cauchy sequence with $\varepsilon = 2014$, there exists N such that

$$|a_j - a_k| < \varepsilon = 2014 \quad \text{whenever } j, k \geq N.$$

Then by the triangle inequality, for $n \geq N$

$$|a_n| \leq |a_n - a_N| + |a_N| \leq 2014 + |a_N|$$

and this gives a bound for the tail $(a_n)_{n\geq N}$. The remaining $N-1$ leading terms are also bounded by

$$|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|\} \quad \text{for } n = 1, \ldots, N-1.$$

Thus if we put $M = \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 2014\}$, we always have $|a_n| \leq M$ and the conclusion follows. □

Exercise 2.5. Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ be Cauchy sequences of rationals. Prove that $(a_n b_n)_{n=0}^{\infty}$ is a Cauchy sequence of rationals. In other words, the multiplication of two real numbers gives another real number. Now, let $(a'_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals that is equivalent to $(a_n)_{n=0}^{\infty}$. Prove that $(a_n b_n)_{n=0}^{\infty}$ is equivalent to $(a'_n b_n)_{n=0}^{\infty}$. In other words, multiplication of real numbers is well-defined.

Solution. For the first part, let $\mathbb{Q} \ni \varepsilon > 0$. Invoking the previous exercise, we can choose $M_0, M_1 \in \mathbb{Q}$ with $|a_n| \leq M_0$ and $|b_n| \leq M_1$ for all $n \in \mathbb{N}$. Choose then $M := \max(M_0, M_1) \in \mathbb{Q}$. Then by definition of M, $|a_n| \leq M$ and $|b_n| \leq M$ for all $n \in \mathbb{N}$. Also by using the definition of Cauchy sequence, there exists a positive natural number N_0 such that $|a_j - a_k| < \varepsilon/2M$ for all $j, k \geq N_0$ and $|b_j - b_k| < \varepsilon/2M$ for all $j, k \geq N_1$. Define then $N := \max(N_0, N_1) \in \mathbb{N}$. By definition of N, we then have $|a_j - a_k| < \varepsilon/2M$ and $|b_j - b_k| < \varepsilon/2M$ for all $j, k \geq N$. By definition of N we then have

$$\begin{aligned} |a_j b_j - a_k b_k| &\leq |a_j||b_j - b_k| + |b_k||a_j - a_k| \\ &< M(\varepsilon/2M) + M(\varepsilon/2M) \\ &= \varepsilon \quad \text{for all } j, k \geq N. \end{aligned}$$

This shows that $(a_n b_n)_{n=0}^{\infty}$ is also a Cauchy sequence of rationals (noting also that the product of rational numbers is a rational number).

The second part also follows in a similar manner. Let $\mathbb{Q} \ni \varepsilon > 0$ be arbitrary. Using the previous exercise, choose $\mathbb{Q} \ni M > 0$ such that $|b_n| < M$ for all $n \in \mathbb{N}$. Then by the definition of equivalence of Cauchy sequences, we can pick a $N \in \mathbb{N}$ such that $|a_n - a'_n| < \varepsilon/M$ whenever $n \geq N$. Then

$$|a_n b_n - a'_n b_n| = |a_n - a'_n||b_n| < (\varepsilon/M)M = \varepsilon \quad \text{for all } n \geq N$$

and hence $(a_n b_n)_{n=0}^{\infty}$ and $(a'_n b_n)_{n=0}^{\infty}$ are equivalent Cauchy sequences. □ □

3. Homework 3

Exercise 3.1. Let x be a real number and let $\varepsilon > 0$ be any rational number. Show that there exists a rational number y such that $|x - y| < \varepsilon$.

Solution. Choose any sequence $(a_n)_{n=0}^{\infty}$ of rationals that represents x, or equivalently, $x = \mathrm{LIM}_{n\to\infty} a_n$. Then for any $\mathbb{Q} \ni \varepsilon > 0$, there exists N such that $|a_j - a_k| < \varepsilon/2$ whenever $j, k \geq N$. This implies that for any $n \geq N$,

$$\begin{aligned} (\forall k \in \mathbb{N}) k \geq N \implies |a_k - a_n| < \varepsilon/2 &\iff (\forall k \in \mathbb{N}) k \geq N \implies a_n - \varepsilon/2 \leq a_k \leq a_n + \varepsilon/2 \\ &\implies \mathrm{LIM}_{k\to\infty}(a_n - \varepsilon/2) \leq \mathrm{LIM}_{k\to\infty} a_k \leq \mathrm{LIM}_{k\to\infty}(a_n + \varepsilon/2) \quad \text{(by Prop 6.30)} \\ &\implies a_n - \varepsilon/2 \leq x \leq a_n + \varepsilon/2 \\ &\implies x - \varepsilon < a_n < x + \varepsilon \\ &\implies |x - a_n| < \varepsilon. \end{aligned}$$

In particular, $|x - a_n| < \varepsilon$ and the choice $y = a_n$ proves the claim. $\square$

Remark. We remark that the prove above also shows the following statement:

Corollary 1. If $x = \mathrm{LIM}_{n\to\infty} a_n$, then for any $\mathbb{Q} \ni \varepsilon > 0$ there exists N such that $|x - a_n| < \varepsilon$ for any $n \geq N$. $\square$

Once we learn the notion of convergence in $\mathbb{R}$, you will readily find that this justifies the notation $\mathrm{LIM}_{n\to\infty}$. Anyway, this will be used to prove Exercise 8.

Exercise 3.2. Let x, z be real numbers with $x < z$. Show that there exists a rational number y with $x < y < z$. (Hint: use the previous exercise, and the Archimedean property.)

Solution. Let $y' = (x + z)/2$. Then $x < y' < z$. By observing that $\varepsilon = (z - x)/2 > 0$, Archimedean property says that we can choose $N \in \mathbb{N}$ satisfying $\frac{1}{N} < \varepsilon$. Then by the previous exercise, we can choose a rational $y \in \mathbb{Q}$ such that $|y' - y| < \frac{1}{N}$. Then

$$x = y' - \varepsilon < y' - \frac{1}{N} < y < y' + \frac{1}{N} < y' + \varepsilon = z.$$

This proves our claim. $\square$

Exercise 3.3. Let x be a real number. Show that there exists a Cauchy sequence of rational numbers $(a_n)_{n=0}^{\infty}$ such that $x = \mathrm{LIM}_{n\to\infty} a_n$, and such that $a_n > x$ for all $n \geq 0$.

Solution. Using the previous exercise, for each $n \in \mathbb{N}$ choose $a_n \in \mathbb{Q}$ such that $x < a_n < x + \frac{1}{n+1}$. Also choose any sequence $(b_n)_{n=0}^{\infty}$ of rationals such that $x = \mathrm{LIM}_{n\to\infty} b_n$. We need to show that $x = \mathrm{LIM}_{n\to\infty} a_n$, or in other words,

(a) $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence, and

(b) $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent.

For (a), let $0 < \varepsilon \in \mathbb{Q}$ be arbitrary and choose N such that $\frac{1}{N+1} < \varepsilon$. Then

$$j, k \geq N \implies x < a_j < x + \frac{1}{j+1} \quad \text{and} \quad x < a_k < x + \frac{1}{k+1}$$
$$\implies -\frac{1}{k+1} < -a_k + x < 0 \implies -\frac{1}{N+1} < x - a_k < 0$$

shows that $(a_n)_{n=0}^{\infty}$ is indeed a Cauchy sequence.

For part (b), let $0 < \varepsilon \in \mathbb{Q}$ be arbitrary. Then by Corollary 1 and the Archimedean property, we can pick N such that $|b_n - x| < \varepsilon/2$ whenever $n \geq N$ and $\frac{1}{N+1} < \varepsilon/2$. Then

$$\begin{aligned} |a_n - b_n| &\leq |a_n - x| + |x - b_n| \\ &< (a_n - x) + (\varepsilon/2) \\ &< \frac{1}{N+1} + (\varepsilon/2) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon. \end{aligned}$$

This proves that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent as desired. $\square$

Exercise 3.4. For every real number x, show that exactly one of the following statements is true: x is positive, x is negative, or x is zero. Show that if x, y are positive real numbers, then $x + y$ is positive, and xy is positive.

Solution. Before the proof, we first claim the following:

Lemma 2. x is positive if and only if there exist a rational $\varepsilon > 0$, a Cauchy sequence $(a_n)_{n=0}^{\infty}$ and a natural number N such that $x = \mathrm{LIM}_{n\to\infty}(a_n)$ and $a_n > \varepsilon$ for all $n \geq N$.

Remark. Note the difference with the definition: In the lemma, only the existence of a such Cauchy sequence is required. On the other hand, the definition needs to hold for *any* Cauchy sequence that represents x.

Proof of Lemma. The ($\Longrightarrow$) direction is clear. So it suffices to prove ($\Longleftarrow$) direction. Let ε, $(a_n)_{n=0}^{\infty}$ and N be as in the Lemma 2. Let $(b_n)_{n=0}^{\infty}$ be any Cauchy sequence with $x = \mathrm{LIM}_{n\to\infty}(b_n)$.

- Since $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent, there exists N_2 such that $|a_n - b_n| < \varepsilon/3$ for any $n \geq N_2$.
- Since $(b_n)_{n=0}^{\infty}$ is Cauchy, there exists N_3 such that $|b_j - b_k| < \varepsilon/3$ for any $j, k \geq N_3$.

Then with $N' := \max\{N, N_2, N_3\}$, we find that

$$n \geq N' \implies b_n = (b_n - b_{N'}) + (b_{N'} - a_{N'}) + a_{N'} > -(\varepsilon/3) - (\varepsilon/3) + \varepsilon = \varepsilon/3.$$

In summary, the definition of positivity for x is satisfied with the positive rational $\varepsilon/3$ (which is independent of the choice of $(b_n)_{n=0}^{\infty}$). This completes the proof. ////

Now we return to the original problem.

- (Trichotomy of ordering) Exactly one of the following holds: x is positive, x is negative, or $x = 0$: To prove this, we check the following three properties: (a) No real number is both positive and negative. (b) 0 is neither positive nor negative. (c) Any non-zero real number is either positive or negative.

Indeed, if we denote the set of positive reals as $\mathcal{P}$, the set of negative reals as $\mathcal{N}$, then these 3 properties show that

$$\mathcal{P} \cap \mathcal{N} = \emptyset, \quad (\mathcal{P} \cup \mathcal{N}) \cap \{0\} = \emptyset, \quad \text{and} \quad \mathbb{R} \setminus \{0\} = \mathcal{P} \cup \mathcal{N}$$

and thus $\mathbb{R}$ is written as the disjoint union of $\mathcal{P}$, $\mathcal{N}$ and $\{0\}$ as desired. So it suffices to prove them.

(a) We prove this by contradiction. Assume otherwise that $x = \mathrm{LIM}_{n\to\infty} a_n$ is both positive and negative. Since $-x = \mathrm{LIM}_{n\to\infty}(-a_n)$, by invoking Definition 6.21, there exists a rational $\varepsilon > 0$ and a natural number $N > 0$ such that

$$a_n > \varepsilon \quad \text{and} \quad -a_n > \varepsilon \quad \text{for any } n \geq N.$$

In particular,

$$a_N > \varepsilon > 0 > -\varepsilon > a_N$$

and this contradicts the trichotomy of rationals. With this claim, it suffices to prove that a real number is non-zero if and only if it is either positive or negative.

(b) Proving that 0 is not positive amounts to showing the negation of Definition 6.21 for $x = 0$:

Claim. For any rational $\varepsilon > 0$, there exists a Cauchy sequence $(a_n)_{n=0}^{\infty}$ with $0 = \mathrm{LIM}_{n\to\infty}(a_n)$ such that for any natural number N, there exists $n > N$ with $a_n \leq \varepsilon$.

But this is clearly satisfied with the choice $(a_n)_{n=0}^{\infty} = (0, 0, 0, \dots)$. The claim that 0 is not negative also follows in the same way.

(c) Choose a rational $\varepsilon > 0$ that satisfies the conclusion of Lemma 6.13. So if $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence with $x = \mathrm{LIM}_{n\to\infty}(a_n)$, then there exist N' such that $|a_n| > \varepsilon$ whenever $n \geq N'$. Also, choose N'' such that $|a_j - a_k| < \varepsilon/3$ for all $j, k \geq N''$. Then for any $n \geq \max\{N', N''\} =: N$, it follows that

$$|a_n| = |(a_n - a_N) - a_N| \geq |a_N| - |a_n - a_N| > \varepsilon - (\varepsilon/3) = 2\varepsilon/3.$$

In particular, $|a_n| > 2\varepsilon/3 > 0$ and hence a_N is either positive or negative. Now if $a_N > 0$, then for any $n \geq N$

$$a_n = a_N + (a_n - a_N) > a_N - |a_n - a_N| > (2\varepsilon/3) - (\varepsilon/3) = \varepsilon/3.$$

and hence x is positive by Lemma 2. If $a_N < 0$ then by applying the similar argument to $-a_N$ shows that $-x$ is positive. Therefore x is either positive or negative as desired.

This completes the proof of trichotomy.

- x and y are positive, then so are $x + y$ and xy.

Suppose that $x = \mathrm{LIM}_{n\to\infty}(a_n)$ and $y = \mathrm{LIM}_{n\to\infty}(b_n)$. Using Definition 6.23, choose $\varepsilon > 0$ and $N > 0$ such that $a_n > \varepsilon$ and $b_n > \varepsilon$ for all $n \geq N$. Then

$$a_n + b_n > 2\varepsilon > 0 \quad \text{and} \quad a_n b_n > \varepsilon^2 > 0 \quad \text{for all } n \geq N.$$

Now by noting that $x+y = \mathrm{LIM}_{n\to\infty}(a_n+b_n)$ and $xy = \mathrm{LIM}_{n\to\infty}(a_n b_n)$, positivity of $x+y$ and xy follow from Lemma 2. □

Exercise 3.5. Let x, y be real numbers. Prove that $(x^2 + y^2)/2 \geq xy$.

Solution. Notice that

$$(x^2 + y^2)/2 \geq xy \iff (x - y)^2 = x^2 - 2xy + y^2 \geq 0.$$

So it suffices to prove that any square of real number is non-negative. Indeed,

- If $z \geq 0$, then it is clear that $z^2 \geq 0$.
- If $z < 0$, then $-z > 0$ and thus $z^2 = (-z)^2 > 0$.

Therefore the claim follows with the choice $z = x - y$. □

Exercise 3.6. Let A be the set of real numbers

$$A = \left\{\frac{1}{n} : n \geq 1,\, n \in \mathbb{N}\right\} = \{1, 1/2, 1/3, 1/4, \ldots\}.$$

Compute $\sup(A)$ and $\inf(A)$.

Solution. We have $\sup(A) = 1$ and $\inf(A) = 0$. For the first statement, note that $x \leq 1$ for all $x \in A$ so 1 is an upper bound for A. Also $1 \in A$, so any $t \in \mathbb{R}$ with $t < 1$ cannot be an upper bound for A. Therefore, 1 is the least upper bound of A. For the second statement, note that $0 \leq x$ for all $x \in A$, so 0 is a lower bound of A. Moreover, for any $t \in \mathbb{R}$ with $t > 0$, there exists a positive natural number n such that $0 < 1/n < t$, by the Archimedean property. That is, any $t > 0$ is not a lower bound for A. That is, 0 is the greatest lower bound of A, so $\inf(A) = 0$. □

4. Homework 4

Before the solution.

How to prove bijectivity? The following equivalence is useful when establishing the bijectivity of a function:

Proposition. Let $f : X \to Y$ be a function. Then the followings are equivalent:

- f is both injective and surjective.
- f has an inverse, i.e., there exists a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Here id_X denotes for the identity function $X \to X$ defined by $\mathrm{id}_X(x) = x$, and likewise for id_Y. The second criteria is often useful when you can explicitly guess how the inverse should be defined.

Exercise 4.1. Show that the notion of two sets having equal cardinality is an equivalence relation. That is, for sets X, Y, Z, show

- X has the same cardinality as X.
- If X has the same cardinality as Y, then Y has the same cardinality as X.
- If X has the same cardinality as Y, and if Y has the same cardinality as Z, then X has the same cardinality as Z.

Proof. Recall the definition that X and Y have the same cardinality if we can find a bijection $X \to Y$.

- (reflexivity) The identity function $\mathrm{id} : X \to X$ (which satisfies $\mathrm{id}(x) = x$ for all $x \in X$) is a bijection from X onto itself. So X has the same cardinality as X, by definition of cardinality. (To see that the identity function is a bijection, note that each $y \in X$ has exactly one element $x \in X$ such that $(id)(x) = y$. That is, we can choose $x := y$ so that $\mathrm{id}(y) = y$ by definition of the identity function. And any other $x \in X$ with $x \neq y$ satisfies $\mathrm{id}(x) = x \neq y$.)
- (symmetry) Suppose there is a bijection $f : X \to Y$. We know that the inverse function $f^{-1} : Y \to X$ is also bijective. Thus Y also have the same cardinality as X, by definition of cardinality.
- (transitivity) Suppose there is a bijection $f : X \to Y$ and $g : Y \to Z$. We claim that $g \circ f : X \to Z$ is also bijective. Indeed, $f^{-1} \circ g^{-1} : Z \to X$ is an inverse of $g \circ f$, which we easily check as follows:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ \mathrm{id}_Y \circ f = f^{-1} \circ f = \mathrm{id}_X$$
$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ f \circ f^{-1} \circ g^{-1} = g \circ \mathrm{id}_Y \circ g^{-1} = g \circ g^{-1} = \mathrm{id}_Z.$$

This proves that $g \circ f$ is bijective and hence X has the same cardinality as X, by definition of cardinality. □

Exercise 4.2. Using a proof by contradiction, show that the set $\mathbb{N}$ of natural numbers is infinite.

Proof. We argue by contradiction. Suppose $\mathbb{N}$ is finite, i.e., there exists $n \in \mathbb{N}$ and there exists a bijection $f : \{1, \ldots, n\} \to \mathbb{N}$. Now take $m = \max\{f(1), \ldots, f(n)\}$. Then clearly $m + 1 \in \mathbb{N}$, but $m + 1$ does not lie in the range $\{f(1), \ldots, f(n)\}$ of f. This is a contradiction to the surjectivity of f! Having found a contradiction, we are done. □

Exercise 4.3. Let X be a subset of the natural numbers $\mathbb{N}$. Prove that X is at most countable.

Proof. Let $X \subseteq \mathbb{N}$. Then either X is finite or infinite. If X is finite, then we are done. So we assume that X is infinite and prove and X is indeed countable.

Idea. Clearly we are going to use the properties of natural number system extensively. We know that $\mathbb{N}$ is ordered in a very nice way: we can count all the natural numbers starting from 0 and adding 1 successively. This property is encoded in the principle of mathematical induction. Consequently, any subset of $\mathbb{N}$ also inherits this ordering structure, which allows us to enumerate all the elements in increasing order. The precise statement that encrypts this idea is as follows:

Theorem (Well-Ordering Principle). Any non-empty subset of $\mathbb{N}$ has a minimum.

This is intuitive clear, but nevertheless requires a rigorous proof. (Interested readers may find a proof by referring Homework 1.1.) In this proof we will actually demonstrate a way of arranging elements of X in increasing order and claim that this arrangement yields a bijection from $\mathbb{N}$ to X.

To this end, we inductively define a sequence $(a_n)_{n=0}^{\infty}$ of natural numbers and a sequence $(X_n)_{n=0}^{\infty}$ of subsets of X as follows:

$$X_0 = \emptyset, \quad a_n = \min(X - X_n), \quad \text{and} \quad X_{n+1} = \{a_0, \dots, a_n\} = X_n \cup \{a_n\}.$$

Then we claim that this yields well-defined sequences with an additional property.

Claim. For every $n \in \mathbb{N}$ the followings hold:
(a) Both $(a_k)_{k=0}^{\infty}$ and $(X_k)_{k=0}^{\infty}$ are well-defined.
(b) $(a_k)_{k=0}^{\infty}$ is (strictly) increasing: $a_0 < a_1 < \cdots < a_n$.

Proof of Claim. We appeal to the principle of mathematical induction.

- (Base case) Clearly $X_0 = \emptyset$ is well-defined. Also, well-ordering principle (WOP) shows that $a_0 = \min X$ exists, hence is well-defined as well and proves (a) for $n = 0$. Part (b) holds trivially.
- (Inductive step) Assume that the claim holds for $n \geq 0$. Then $X_{n+1} = X_n \cup \{a_n\}$ is also well-defined.

  Next, we check that a_{n+1} is well-defined. By invoking WOP again, this amounts to prove that $X - X_{n+1}$ is non-empty, or in other words, $X \neq X_{n+1}$. Indeed, (b) shows that the mapping
  $$\{1, \dots, n+1\} \to X, \quad k \mapsto a_{k-1}$$
  is injective with the range X_{n+1}. Thus if $X = X_{n+1}$ then this map is also surjective and hence bijective. This contradicts the assumption that X is infinite. Therefore $X \neq X_{n+1}$ and the well-definedness follows. This proves part (a) for $n+1$.

  For (b), $X - X_{n+1} \subseteq X - X_n$ shows that
  $$a_n = \min(X - X_n) \leq \min(X - X_{n+1}) = a_{n+1}.$$
  But if $a_n = a_{n+1}$, then we have both $a_n = a_{n+1} \in X - X_{n+1}$ and $a_n \in X_{n+1}$, a contradiction! So we must have $a_n < a_{n+1}$ and part (b) follows for $n+1$.

  Therefore by induction the claim follows for all $n \in \mathbb{N}$. ////

Now we are ready to prove the countability of X. Define $f : \mathbb{N} \to X$ by $f(n) = a_n$. Then

- f is injective by (b) of the claim. Indeed, if $m \neq n$, then by assuming $m < n$ without losing the generality we have $f(m) = a_m < a_n = f(n)$ and hence $f(m) \neq f(n)$.
- f is surjective. To this end, suppose otherwise. Then $X - f(\mathbb{N})$ is non-empty and we can pick the minimum $m = \min(X - f(\mathbb{N}))$. Then $X - f(\mathbb{N}) \subseteq X - X_n$ shows that
  $$a_n = \min(X - X_n) \leq \min(X - f(\mathbb{N})) = m \quad \text{for any } n \in \mathbb{N}.$$

This shows that the sequence

$$(m - a_0, m - a_1, m - a_2, \dots)$$

is a (strictly) decreasing sequence of natural number, a contradiction by infinite descent. This proves that f is surjective.

Therefore f gives a bijection from $\mathbb{N}$ to X and hence X is countable. $\square$

Exercise 4.4. Let Y be a set. Let $f\colon \mathbb{N} \to Y$ be a function. Then $f(\mathbb{N})$ is at most countable. (Hint: consider the set $A := \{n \in \mathbb{N}\colon f(n) \neq f(m) \text{ for all } 0 \leq m < n\}$. Prove that f is a bijection from A onto $f(\mathbb{N})$. Then use the previous exercise.)

Proof. Following the hint, we show that the function $f|_A : A \to f(\mathbb{N})$ which is induced from f by $f|_A(n) = f(n)$ is bijective.

- $f|_A$ is injective. Indeed, let $m, n \in A$ be distinct. We may assume $m < n$ without losing the generality. Then by definition of A,
$$f|_A(m) = f(m) \neq f(n) = f|_A(n)$$
and hence $f|_A$ is injective.
- $f|_A$ is surjective. To this end, pick any $y \in f(\mathbb{N})$. Then the set
$$X = \{n \in \mathbb{N} : f(n) = y\}$$
is non-empty, and thus by WOP we can pick the minimum
$$n = \min X.$$

Then for any $0 < m < n$ we must have $f(m) \neq y = f(n)$, for otherwise we get a contradiction to the minimality of n. This shows that $n \in A$ and $f|_A(n) = f(n) = y$, hence $f|_A$ is surjective.

Therefore $f|_A$ gives a bijection from A to $f(\mathbb{N})$. But since we know that A is at most countable by the previous exercise, the same is true for $f(\mathbb{N})$ (by the first exercise). $\square$

Exercise 4.5. Let X, Y be countable sets. Show that $X \cup Y$ is a countable set.

Solution. Let $f\colon X \to \mathbb{N}$ be a bijection. Let $g\colon Y \to \mathbb{N}$ be a bijection. We need to find a bijection $h\colon X \cup Y \to \mathbb{N}$. We define h as follows. $h(x) := 2f(x)$ for all $x \in X$ and $h(y) := 2h(y) + 1$ for all $y \in Y$.

Proof of surjectivity. For any $n \in \mathbb{N}$, either n is even or odd. If n is even, $n/2 \in \mathbb{N}$. Since f is a bijection, there exists $x \in X$ such that $f(x) = n/2$, i.e. $2f(x) = n$, i.e. $h(x) = n$. If n is odd, then $(n-1)/2 \in \mathbb{N}$. Since g is a bijection, there exists $y \in Y$ such that $g(y) = (n-1)/2$, i.e. $2g(y) + 1 = n$, i.e. $h(y) = n$. So, h is surjective.

Proof of injectivity. Let $a, b \in X \cup Y$. Assume $h(a) = h(b)$. If $a, b \in X$, then $h(a) = 2f(a)$ and $h(b) = 2f(b)$ by definition of h, so that $2f(a) = 2f(b)$, i.e. $f(a) = f(b)$, so that $a = b$ by injectivity of f. Similarly, if $a, b \in Y$, then $h(a) = 2g(a)+1$ and $h(b) = 2g(b)+1$ by definition of h, so that $2g(a)+1 = 2g(b)+1$, i.e. $g(a) = g(b)$, so that $a = b$ by injectivity of g. In the remaining case that a, b are not both in X or both in Y, we have $a \in X$ and $b \in Y$ (without loss of generality), in which case $h(a)$ is even by definition of h, and $h(b)$ is odd by definition of h, which violates that $h(a) = h(b)$, i.e. this last case cannot occur.

Since we have shown that h is surjective and injective, we conclude that h is a bijection. $\square$

Exercise 4.6. Let X, Y be countable sets. Show that $X \times Y$ is a countable set.

Solution. This follows from Lemma 2.1.22 in the notes, where we showed that $\mathbb{N} \times \mathbb{N}$ is countable. Since X and Y are each countable, we can let $f\colon X \to \mathbb{N}$ be a bijection and let $g\colon Y \to \mathbb{N}$ be a bijection. We then claim that $h\colon X \times Y \to \mathbb{N} \times \mathbb{N}$ defined by $h(x, y) := (f(x), g(y))$ is a bijection.

Proof of surjectivity. Let $m, n \in \mathbb{N}$. Since f, g are each bijections, we can find $x \in X$ and $y \in Y$ such that $f(x) = m$ and $g(y) = n$. So, by definition of h we have $h(x, y) = (f(x), g(y)) = (m, n)$. So, h is surjective.

Proof of injectivity. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Assume $h(x_1, y_1) = h(x_2, y_2)$. By definition of h, this means $f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$. Since f, g are each injective themselves, we conclude that $x_1 = x_2$ and $y_1 = y_2$. That is, $(x_1, y_1) = (x_2, y_2)$. We have shown therefore that h is injective.

Since we have shown that h is surjective and injective, we conclude that h is a bijection.

Since h is a bijection from $X \times Y$ to $\mathbb{N} \times \mathbb{N}$, we conclude by definition of cardinality that $X \times Y$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality. Since $\mathbb{N} \times \mathbb{N}$ is countable by Lemma 2.1.22 in the notes, we conclude that $X \times Y$ is also countable. □

5. Homework 5

Exercise 5.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is convergent if and only if $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence. (Hint: Given a Cauchy sequence $(a_n)_{n=0}^{\infty}$, use that the rationals are dense in the real numbers to replace each real a_n by some rational a'_n, so that $|a_n - a'_n|$ is small. Then, ensure that the sequence $(a'_n)_{n=0}^{\infty}$ is a Cauchy sequence of rationals and that $(a'_n)_{n=0}^{\infty}$ defines a real number which is the limit of the original sequence $(a_n)_{n=0}^{\infty}$.)

Preliminary Before the solution, we remark the following observation (which you may already know if you have carefully read my solution of 2nd homework):

Observation 1. *Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rational numbers and $x = LIM_{n\to\infty}(a_n) \in \mathbb{R}$ its formal limit. Then $(a_n)_{n=0}^{\infty}$ actually converges to x as a sequence in $\mathbb{R}$. In other words,*

$$x = \mathrm{LIM}_{n\to\infty}(a_n) \implies x = \lim_{n\to\infty} a_n.$$

Proof of Observation 1. Before the proof, we remind that what we want to prove is the following statement:

$$(\forall \varepsilon \in \mathbb{R})\ \varepsilon > 0 \implies ((\exists N \in \mathbb{N})(\forall n \in \mathbb{N})\ n \geq N \implies |a_n - x| < \varepsilon).$$

To this end, choose arbitrary positive real $\varepsilon > 0$ and pick a positive rational ε' with $0 < \varepsilon' < \varepsilon$ by utilizing the Archimedean property. Then there exists N (depending on ε' and hence on ε) such that

$$\begin{aligned} j, k \geq N &\implies |a_j - a_k| < \varepsilon' \\ k \geq N &\implies |a_n - a_k| < \varepsilon'. \end{aligned}$$

Now fix $n \geq N$. Then we get In view of Proposition 6.30, taking formal limit $\mathrm{LIM}_{k\to\infty}$, we obtain

$$|a_n - x| \leq \varepsilon'.$$

In summary, for any positive real $\varepsilon > 0$ we was able to find $N \in \mathbb{N}$ such that $|a_n - x| < \varepsilon$ whenever $n \geq N$. This proves the desired statement at the beginning of the proof and hence a_n converges to x in $\mathbb{R}$. /////

Solution. With this observation in our hand, the solution is as follows:

- (⇒): Assume that $(a_n)_{n=0}^{\infty}$ is convergent with the limit $L \in \mathbb{R}$. To prove that $(a_n)_{n=0}^{\infty}$ is Cauchy, we invoke the standard '2ε-argument'. For any positive real $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < \frac{\varepsilon}{2}.$$

Then for any $j, k \geq N$, the triangle inequality shows

$$|a_j - a_k| \leq |a_j - L| + |L - a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $(a_n)_{n=0}^{\infty}$ is Cauchy.

- ($\Leftarrow$): Assume that $(a_n)_{n=0}^{\infty}$ is Cauchy.

  Idea. If it were the case that $(a_n)_{n=0}^{\infty}$ consists of only rational numbers, then $(a_n)_{n=0}^{\infty}$ would have become convergent in view of Observation 1. But here $(a_n)_{n=0}^{\infty}$ is assumed to be any Cauchy sequence of real numbers. To remedy this situation, we approximate $(a_n)_{n=0}^{\infty}$ by a sequence of rational numbers, check that this approximation converges, and finally the original sequence also converges to the same limit.

  (Step 1) To realize this idea, we first choose an "approximating sequence". Define a sequence $(a'_n)_{n=0}^{\infty}$ of rational numbers by picking $a'_n \in \mathbb{Q}$ satisfying

$$|a_n - a'_n| < \frac{1}{n+1}$$

for each $n \in \mathbb{N}$. (This is possible by Exercise 6 in Homework 2.) We claim the followings:

(1) (a'_n) is a Cauchy sequence in the sense of Definition 5.6.
(2) If we put $x = \mathrm{LIM}_{n\to\infty}(a'_n)$, then $(a'_n)_{n=0}^{\infty}$ converges to x.

Note that the statement follows once (1) and (2) are verified.

  (Step 2) To prove (1), let $\varepsilon > 0$ be any positive rational. We invoke the standard '3ε-argument' as follows:

  – By exploiting the Archimedean property, pick $N_1 \in \mathbb{N}$ such that $1/(N_1 + 1) < (\varepsilon/3)$.
  – Since $(a_n)_{n=0}^{\infty}$ is Cauchy, pick $N_2 \in \mathbb{N}$ such that $|a_j - a_k| < (\varepsilon/3)$ whenever $j, k \geq N_2$.

Then for $N = \max\{N_1, N_2\}$, it follows from the triangle inequality that

$$\begin{aligned} j, k \geq N \implies |a'_j - a'_k| &= |a'_j - a_j + a_j - a_k + a_k - a'_k| \\ &\leq |a'_j - a_j| + |a_j - a_k| + |a_k - a'_k| \\ &< \frac{1}{j+1} + \frac{\varepsilon}{3} + \frac{1}{k+1} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves that $(a'_n)_{n=0}^{\infty}$ is Cauchy in the sense of Definition 5.6. Then by Observation 1, $(a'_n)_{n=0}^{\infty}$ converges to the formal limit $x = \mathrm{LIM}_{n\to\infty}(a'_n)$. (Step 3) To complete the proof, we prove that $(a_n)_{n=0}^{\infty}$ also converges to x. This essentially follows from the squeezing lemma, which is not in our hand yet. So we make a direct proof with the standard '2ε-argument' as follows: Let $\varepsilon > 0$ be an arbitrary positive real. Then

  – Choose $N_1 \in \mathbb{N}$ such that $1/(N_1 + 1) < (\varepsilon/2)$ with aid of the Archimedean property.
  – Choose N_2 such that $|a'_n - x| < (\varepsilon/2)$ whenever $n \geq N_2$.

Then for $N = \max\{N_1, N_2\}$, the triangle inequality says

$$n \geq N \implies |a_n - x| \leq |a_n - a'_n| + |a'_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof that $(a_n)_{n=0}^{\infty}$ is convergent as desired.

□

Exercise 5.2. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be real convergent sequences. Let x, y be real numbers such that $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$.

(i) The sequence $(a_n + b_n)_{n=0}^{\infty}$ converges to $x + y$. That is,

$$\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n).$$

(ii) The sequence $(a_n b_n)_{n=0}^{\infty}$ converges to xy. That is,

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

(iii) For any real number c, the sequence $(ca_n)_{n=0}^{\infty}$ converges to cx. That is,

$$c \lim_{n\to\infty} a_n = \lim_{n\to\infty}(ca_n).$$

(iv) The sequence $(a_n - b_n)_{n=0}^{\infty}$ converges to $x - y$. That is,

$$\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n).$$

(v) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(a_n^{-1})_{n=m}^{\infty}$ converges to x^{-1}. That is,

$$\lim_{n\to\infty} a_n^{-1} = (\lim_{n\to\infty} a_n)^{-1}.$$

(vi) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(b_n/a_n)_{n=m}^{\infty}$ converges to y/x. That is,

$$\lim_{n\to\infty}(b_n/a_n) = (\lim_{n\to\infty} b_n)/(\lim_{n\to\infty} a_n).$$

(vii) Suppose $a_n \geq b_n$ for all $n \geq 0$. Then $x \geq y$.

(Hint: you can save time by using some of these statements to prove the others. For example: (iii) follows from (ii); (iv) follows from (i); and (vi) follows from (v) and (ii).)

Solution.

(i) We invoke the standard '2ε-argument'. Let $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad n \geq N_2 \implies |b_n - y| < \frac{\varepsilon}{2}.$$

Then for $N = \max\{N_1, N_2\}$, we obtain from the triangle inequality that

$$\begin{aligned} n \geq N \implies |(a_n + b_n) - (x + y)| &\leq |a_n - x| + |b_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that $\lim_{n\to\infty}(a_n + b_n) = x + y$ as expected.

(ii) We invoke the following variant of the '2ε-argument': From Corollary 3.14, we know that $(a_n)_{n=0}^\infty$ is bounded. Pick a bound $M > 0$ of $(a_n)_{n=0}^\infty$. Now let $\varepsilon > 0$ be arbitrary, and choose $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - x| < \frac{\varepsilon}{2(|y| + 2014)} \quad \text{and} \quad n \geq N_2 \implies |b_n - y| < \frac{\varepsilon}{2M}.$$

Then for $N = \max\{N_1, N_2\}$, we obtain from the triangle inequality that

$$\begin{aligned} n \geq N \implies |a_n b_n - xy| &= |a_n b_n - a_n y + a_n y - xy| \\ &\leq |a_n||b_n - y| + |a_n - x||y| \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2(|y| + 2014)} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(Notice here how we circumvented the technical issues of possible division by zero.) This completes the proof.

(iii) Notice that the constant sequence $(b_n)_{n=0}^\infty = (c, c, c, \dots)$ converges to c. (For any $\varepsilon > 0$, just pick $N = 0$ for the definition of convergence.) Plug this to (ii).

(iv) Utilize (iii) with the choice $c = -1$ to derive that $\lim_{n\to\infty}(-b_n) = -y$. Then apply (i) to $(a_n)_{n=0}^\infty$ and $(-b_n)_{n=0}^\infty$.

(v) The key ingredient is to find a lower bound of $(a_n)_{n=m}^\infty$. Indeed, with the choice $\varepsilon = |x|/2$ there exists $N \in \mathbb{N}$ such that $|a_n - x| < \varepsilon = |x|/2$ whenever $n \geq N$. Then by the reverse triangle inequality,

$$n \geq N \implies |a_n| = |x + (a_n - x)| \geq |x| - |a_n - x| > |x| - \frac{|x|}{2} = \frac{|x|}{2}.$$

Thus if we put $\delta = \min\{|a_m|, |a_{m+1}|, \dots, |a_{N-1}|, |x|/2\}$ then we have $\delta > 0$ by the assumption and

$$(\forall n \in \mathbb{N}) \quad n \geq m \implies |a_n| \geq \delta. \qquad (1)$$

This lower bound is necessary for the actual proof as we will see.

Returning to the actual proof, pick any $\varepsilon > 0$. (Forget the choice of ε above!) Then choose $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - x| < \delta\varepsilon|x|.$$

Then it follows from (1) that

$$n \geq N \implies |a_n| \geq \delta \quad \text{and} \quad |a_n - x| < \delta\varepsilon|x| \implies \left|\frac{1}{a_n} - \frac{1}{x}\right| = \frac{|a_n - x|}{|a_n||x|} < \frac{\delta\varepsilon|x|}{\delta|x|} = \varepsilon.$$

This proves that $(a_n^{-1})_{n=m}^\infty$ converges to x^{-1}.

(vi) Apply (v) to deduce that $(b_n^{-1})_{n=m}^\infty$ converges to y^{-1}. Then apply (ii) to $(a_n)_{n=m}^\infty$ and $(b_n^{-1})_{n=m}^\infty$.

(vii) Assume otherwise so that $x < y$. Then for the choice $\varepsilon = (y - x)/3$, we can find $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - x| < \varepsilon \quad \text{and} \quad n \geq N_2 \implies |b_n - y| < \varepsilon.$$

Choose any $n \geq \max\{N_1, N_2\}$. Then

$$x - \varepsilon < a_n \quad \text{and} \quad b_n < y + \varepsilon$$

and hence we get

$$\frac{y-x}{3} = \varepsilon < b_n - y \quad \text{and} \quad x - \varepsilon < a_n \implies \frac{y-x}{3} = \varepsilon < 2\varepsilon$$

a contradiction! Therefore $x \geq y$ as desired.

□

Exercise 5.3. For each natural number n, let a_n be a real number such that $|a_n| \leq 2^{-n}$. Define $b_n := a_1 + a_2 + \cdots + a_n$. Prove that the sequence $(b_n)_{n=0}^{\infty}$ is convergent.

Solution. This is a special case of Proposition 7.15. The idea is that, since it is almost impossible to find an expression for a possible limit of (b_n), we appeal to an indirect argument. That is, we show that (b_n) is Cauchy. Once this is established, the completeness of $\mathbb{R}$ guarantees the existence of a limit.

For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. (Indeed, using the Archimedean property choose $N \in \mathbb{N}$ such that $N\varepsilon > 1$. Now check that $k \leq 2^k$ for any $k \in \mathbb{N}$ either from mathematical induction or from any appropriate theorem you like. This gives $1 < 2^N\varepsilon$ and hence $2^{-N} < \varepsilon$.) Also let $j, k \geq N$ be arbitrary. Without loss of generality we assume that $j \geq k$. Then

$$\begin{aligned}
|b_j - b_k| &= |a_{k+1} + \cdots + a_j| \\
&\leq |a_{k+1}| + \cdots + |a_j| \\
&< 2^{-(k+1)} + \cdots + 2^{-j} \\
&= 2^{-k}(2^{-1} + \cdots + 2^{-(j-k)}) \\
&= 2^{-k}\frac{1 - 2^{-(j-k)}}{2} < 2^{-k} \leq 2^{-N} < \varepsilon.
\end{aligned}$$

This shows that (b_n) is Cauchy, hence convergent. This completes the proof. □

Exercise 5.4. Let E be a subset of $\mathbb{R}^*$. Then the following statements hold.

- For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$.
- Let $M \in \mathbb{R}^*$ be an upper bound for E, so that $x \leq M$ for all $x \in E$. Then $\sup(E) \leq M$.
- Let $M \in \mathbb{R}^*$ be a lower bound for E, so that $x \geq M$ for all $x \in E$. Then $\inf(E) \geq M$.

(Hint: it may be helpful to break into cases concerning whether or not E contains $+\infty$ or $-\infty$.)

Solution.

- For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$:
  The proof is just a typical application of the divide-and-conquer method. We first prove that $x \leq \sup(E)$.
  - (Case 1) Suppose that $\emptyset \neq E \subseteq \mathbb{R}$ and is bounded above. Then $\sup(E) \in \mathbb{R}$ is the least upper bound of E and thus $x \leq \sup(E)$ for all $x \in E$ by the definition of upper bound.

- (Case 2) Suppose that $\emptyset \neq E \subseteq \mathbb{R}$ but is not bounded above. Then $\sup(E)$ is defined as $+\infty$, and by the definition of ordering on $\mathbb{R}^*$ we always have $x \leq +\infty = \sup(E)$ for all $x \in E$.
- (Case 3) Suppose that $E = \emptyset$. Then the statement $(\forall x \in E)\ x \in E \implies x \leq \sup(E)$ is vacuously true.
- (Case 4) Suppose that $+\infty \in E$. Then $\sup(E)$ is defined as $+\infty$, and the claim follows exactly by the same argument as above.
- (Case 5) Suppose that $+\infty \notin E$ and $-\infty \in E$. Then $\sup(E) := \sup(E \backslash \{-\infty\})$. So if $x \in E$, then either $x \in E \backslash \{-\infty\}$ or $x = -\infty$. In the former case, $x \leq \sup(E)$ follows from one of Case 1-3. (Note that any of Case 1-3 is possible.) And in the latter case, $x \leq \sup(E)$ follows from the definition of ordering on $\mathbb{R}^*$.

These five cases exhaust all the possible cases for $E \subseteq \mathbb{R}^*$. The proof for $x \geq \inf(E)$ follows mutatis mutandis.

- Let $M \in \mathbb{R}^*$ be an upper bound for E, so that $x \leq M$ for all $x \in E$. Then $\sup(E) \leq M$. Again, we prove this by dividing cases:
  - (Case 1) If either $M = +\infty$ or $\sup(E) = -\infty$, then in view of the definition of ordering on $\mathbb{R}^*$, there is nothing to prove.
  - (Case 2) If $M = -\infty$, then whenever $x \in E$ we must have $-\infty \leq x \leq M = -\infty$ and thus $x = -\infty$. This shows that either $E = \emptyset$ or $E = \{-\infty\}$. In any cases, we have $\sup(E) = -\infty$ and hence $\sup(E) \leq M$.
  - (Case 3) So it suffices to show the claim when $M \in \mathbb{R}$ and $\sup(E) > -\infty$. An immediate observation is that $+\infty \notin E$ and $E \backslash \{-\infty\} \subseteq \mathbb{R}$ is non-empty.
    * If $-\infty \notin E$, then $E \subseteq \mathbb{R}$ is non-empty and M is an upper bound of E. So $\sup(E) \leq M$ is clear.
    * If $-\infty \in E$, then $E \backslash \{-\infty\} \subseteq \mathbb{R}$ is non-empty and M is an upper bound of $E \backslash \{-\infty\}$. So $\sup(E) = \sup(E \backslash \{-\infty\}) \leq M$.

  This proves the desired claim.
- This follows from the same argument as in the previous part.

□

Exercise 5.5. Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers. Let x be the extended real number $x := \sup(a_n)_{n=m}^\infty$. Then $a_n \leq x$ for all $n \geq m$. Also, for any $M \in \mathbb{R}^*$ which is an upper bound for $(a_n)_{n=m}^\infty$ (so that $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for any $y \in \mathbb{R}^*$ such that $y < x$, there exists at least one integer n with $n \geq m$ such that $y < a_n \leq x$. (Hint: use the previous exercise.)

Solution.

- For any $n \geq m$, $x = \sup\{a_n : n \geq m, n \in \mathbb{N}\}$ is an upper bound of the set $\{a_n : n \geq m, n \in \mathbb{N}\}$ in which a_n is contained. So we have $a_n \leq x$.
- If M is an upper bound of $(a_n)_{n=m}^\infty$, then it is also an upper bound of the set $\{a_n : n \geq m, n \in \mathbb{N}\}$. So by the previous exercise we get $x = \sup\{a_n : n \geq m, n \in \mathbb{N}\} \leq M$.
- Finally, assume otherwise. Then for any $n \geq m$ we have either $y \geq a_n$ or $a_n > x$. Since the latter is impossible, we always have $y \geq a_n$ for all $n \geq m$. Then y is an upper bound of $(a_n)_{n=m}^\infty$ and hence $y \geq x$, contradicting the assumption. This completes the proof.

□

Exercise 5.6. Let $(a_n)_{n=m}^{\infty}$ be a bounded sequence of real numbers. Assume also that $(a_n)_{n=m}^{\infty}$ is monotone increasing. That is, $a_{n+1} \geq a_n$ for all $n \geq m$. Then the sequence $(a_n)_{n=m}^{\infty}$ is convergent. In fact,

$$\lim_{n\to\infty} a_n = \sup (a_n)_{n=m}^{\infty}.$$

(Hint: use the previous exercise.)

Solution. Let $\varepsilon > 0$ be an arbitrary positive real number. Denote $x = \sup (a_n)_{n=m}^{\infty}$. Since $(a_n)_{n=m}^{\infty}$ is bounded, we have $x \in \mathbb{R}$. Then $x - \varepsilon < x$ and thus we can choose an integer $N \geq m$ such that $x - \varepsilon < a_N \leq x$. Then for any $n \geq N$, we have

$$x - \varepsilon < a_N \leq a_{N+1} \leq \cdots \leq a_n \leq x < x + \varepsilon$$

and hence $|a_n - x| < \varepsilon$. This proves that $(a_n)_{n=m}^{\infty}$ converges to x. □

6. Homework 6

Exercise 6.1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers that converges to a real number x. Then x is a limit point of $(a_n)_{n=m}^{\infty}$. Moreover, x is the only limit point of $(a_n)_{n=m}^{\infty}$.

Solution.

- The first part is essentially follows by the definition: Let $\varepsilon > 0$ and $N > m$ be arbitrary. We have to show that $|a_n - x| < \varepsilon$ holds for some $n \geq N$.
  Indeed, from the convergence of $(a_n)_{n=m}^{\infty}$, there exists $N' \in \mathbb{N}$ such that we have $|a_n - x| < \varepsilon$ for any $n \geq N'$. Then by picking any $n \in \mathbb{N}$ that satisfies $n \geq \max\{N, N'\}$, the definition of limit point is satisfied.
- It amounts to proving that $(a_n)_{n=m}^{\infty}$ has a unique limit point, which is x. Assume that $y \in \mathbb{R}$ is any limit point of $(a_n)_{n=m}^{\infty}$. To show that $y = x$, we assume otherwise and derive a contradiction. To this end, for $\varepsilon = |y - x|/3 > 0$,
  - Pick N_1, from the convergence of $(a_n)_{n=m}^{\infty}$, such that $|a_n - x| < \varepsilon$ whenever $n \geq N_1$.
  - Pick $n \geq N_1$, from the definition of limit point for y, such that $|a_n - y| < \varepsilon$.

  Then it follows that

  $$3\varepsilon = |y - x| = |(y - a_n) + (a_n - x)| \leq |y - a_n| + |a_n - x| < \varepsilon + \varepsilon = 2\varepsilon,$$

  a contradiction! Therefore $y = x$ as desired and the claim follows.

□

Exercise 6.2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence. (Note that $L^+, L^- \in \mathbb{R}^*$.)

(iii) $\inf (a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup (a_n)_{n=m}^{\infty}$.
(iv) If c is any limit point of $(a_n)_{n=m}^{\infty}$, then $L^- \leq c \leq L^+$.
(v) If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. If L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.
(vi) Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.

Solution. Before the proof, define sequences $(\ell_n)_{n=m}^{\infty}$ and $(u_n)_{n=m}^{\infty}$ in $\mathbb{R}^*$ by

$$\ell_n = \inf{}_{k\geq n}(a_k) \quad \text{and} \quad u_n = \sup{}_{k\geq n}(a_k) \quad \text{for } n \geq m,$$

respectively. This will save our ink and space. We also remark that $(\ell_n)_{n=m}^{\infty}$ is monotone increasing and $(u_n)_{n=m}^{\infty}$ is monotone decreasing in $\mathbb{R}^*$, which we have already checked in Definition 6.4 and 6.5.

(iii) It is clear that

$$\inf{}_{k\geq m}(a_k) = \ell_m \leq \ell_n \leq a_n \leq u_n \leq u_m = \sup{}_{k\geq m}(a_k) \quad \text{for all } n \geq m.$$

Now the first inequality and the third inequality are immediate:

$$\ell_m \leq \sup{}_n(\ell_n) = L^- \quad \text{and} \quad L^+ = \inf{}_n(u_n) \leq u_m.$$

To prove the intermediate inequality, notice that every u_k is an upper bound of $(\ell_n)_{n=m}^{\infty}$ and likewise that every ℓ_j is a lower bound of $(u_n)_{n=m}^{\infty}$. Indeed, let $j, k \geq m$ be arbitrary. Then

- If $j \geq k$, then $\ell_j \leq u_j \leq u_k$.
- If $j \leq k$, then $\ell_j \leq \ell_k \leq u_k$.

So in any cases $\ell_j \leq u_k$ for any $j, k \geq m$ and the claim follows. Now using Proposition 5.4 (or equivalently, Exercise 5), we have

$$L^- = \sup{}_n(\ell_n) \leq u_k \quad \text{for all } k \geq m$$

and now taking $\inf{}_{k\geq m}$ we finally get

$$L^- \leq L^+$$

as desired. *(Remark.* In summary, we combined the pairwise estimate $\ell_n \leq u_n$ and the monotonicity of (ℓ_n) and (u_n) to obtain the stronger estimate $\ell_j \leq u_k$. This allows us to take infimum and supremum separately for j and k. This kind of trick will appear again, especially when we learn the Riemann integral.)

(iv) We first focus on $c \leq L^+$. If $L^+ = +\infty$, there is nothing to prove. So we assume that $L^+ < +\infty$. To this end we invoke the standard '2ε-argument': Let $\varepsilon > 0$ be arbitrary. Then

- Since $\lim_{n\to\infty} u_n = L^+$, there exists N such that $|u_n - L^+| < (\varepsilon/2)$ whenever $n \geq N$.
- Since c is a limit point, there exists $n \geq N$ such that $|a_n - c| < (\varepsilon/2)$.

Then with the choice of n above, we get

$$c = (c - a_n) + (a_n - L^+) + L^+ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + L^+ = L^+ + \varepsilon.$$

Now the resulting inequality

$$c < L^+ + \varepsilon$$

depends only on $\varepsilon > 0$. Since ε is arbitrary, this implies that $c \leq L^+$. The proof for $L^- \leq c$ follows in exactly the same way.

(v) Assume that L^+ is finite. This means that $(u_n)_{n=m}^{\infty}$ converges to L^+. Now let $\varepsilon > 0$ and $N \in \mathbb{N}$ be arbitrary with $N \geq m$. We want to prove that $|a_n - L^+| < \varepsilon$ for some $n \geq N$.

Indeed, choose $N' \in \mathbb{N}$ such that $|u_n - L^+| < \varepsilon$ for any $n \geq N'$. Also pick any $m \geq \max\{N, N'\}$. Then by noticing that

$$L^- - \varepsilon < u_m < L^+ + \varepsilon \quad \text{and} \quad u_m = \sup{}_{k \geq m}(a_k),$$

it follows from Proposition 5.4 (or equivalently, Exercise 5 above) that there exists $n \geq m$ satisfying

$$L^+ - \varepsilon < a_n \leq \sup{}_{k \geq m}(a_k) = u_m < L^+ + \varepsilon.$$

This inequality shows that $|a_n - L^+| < \varepsilon$. In summary, what we have shown so far is: for any $\varepsilon > 0$ and $N \in \mathbb{N}$ with $N \geq m$, there exists $n \geq N$ such that $|a_n - L^+| < \varepsilon$. This shows that L^+ is a limit point of $(a_n)_{n=m}^{\infty}$. The proof for L^- also follows in exactly the same way.

(vi) $(\Rightarrow)$: If $(a_n)_{n=m}^{\infty}$ converges to c, then by Proposition 6.2 (or equivalently, Exercise 6 above) shows that c is the unique limit point of $(a_n)_{n=m}^{\infty}$. Also, since $(a_n)_{n=m}^{\infty}$ is bounded, both L^+ and L^- are finite. Then by (v) above, L^+ and L^- are limit points of $(a_n)_{n=m}^{\infty}$. Therefore by the uniqueness we get $L^+ = L^- = c$. (Of course, a direct proof is also possible. Try it yourself!)

$(\Leftarrow)$: We know that $u_n \to L^+ = c$ and $l_n \to L^- = c$. Now for any $\varepsilon > 0$, choose $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |u_n - c| < \varepsilon \quad \text{and} \quad n \geq N_2 \implies |l_n - c| < \varepsilon.$$

Then with $N = \max\{N_1, N_2\}$, it follows that

$$n \geq N \implies c - \varepsilon < l_n \leq a_n \leq u_n < c + \varepsilon \implies |a_n - c| < \varepsilon.$$

This proves that $a_n \to c$ as $n \to \infty$.

□

Exercise 6.3. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers such that $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite. Prove:

$$\limsup_{n\to\infty}(a_n + b_n) \leq (\limsup_{n\to\infty} a_n) + (\limsup_{n\to\infty} b_n).$$

Solution. Let $u_n = \sup{}_{k\geq n}(a_k)$ and $v_n = \sup{}_{k\geq n}(b_k)$ be suprema. Then the following relation follows by the definition of supremum:

$$a_k \leq u_n \quad \text{and} \quad b_k \leq v_n, \quad \forall k \geq n \implies a_k + b_k \leq u_n + v_n, \quad \forall k \geq n$$

shows that, upon taking the supremum over all k with $k \geq n$, we get

$$\sup{}_{k\geq n}(a_k + b_k) \leq u_n + v_n.$$

This shows that for any $n \geq m$, we get

$$\inf{}_{j:j\geq m} \sup{}_{k\geq j}(a_k + b_k) \leq \sup{}_{k\geq n}(a_k + b_k) \leq u_n + v_n.$$

Since we know that $(u_n)_{n=m}^{\infty}$ and $(v_n)_{n=m}^{\infty}$ converge to $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ respectively, taking $n \to \infty$ gives

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

as desired. □

Exercise 6.4. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers. Assume that $a_n \le b_n$ for all $n \ge m$. Prove:

- $\sup (a_n)_{n=m}^{\infty} \le \sup (b_n)_{n=m}^{\infty}$.
- $\inf (a_n)_{n=m}^{\infty} \le \inf (b_n)_{n=m}^{\infty}$.
- $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$.

Solution.

- Let k and n be any integer satisfying $k \ge n \ge m$. Then the following obvious relations
$$a_k \le b_k \quad \text{and} \quad b_k \le \sup{}_{j\ge n}(b_j)$$
shows that $a_k \le \sup{}_{j\ge n}(b_j)$ for all $k \ge n$. In particular, the supremum $\sup{}_{j\ge n}(b_j)$ is an upper bound of the set $\{a_n, a_{n+1}, \dots\}$ and hence we get
$$\sup{}_{j\ge n}(a_j) = \sup\{a_n, a_{n+1}, \dots\} \le \sup{}_{j\ge n}(b_j).$$
- This follows from exactly the same argument. Indeed, the following proof is just a duplication of the previous proof:
  Let k and n be any integer satisfying $k \ge n \ge m$. Then the following obvious relations
$$a_k \le b_k \quad \text{and} \quad \inf{}_{j\ge n}(a_j) \le a_k$$
shows that $\inf{}_{j\ge n}(a_j) \le b_k$ for all $k \ge n$. In particular, the infimum $\inf{}_{j\ge n}(a_j)$ is a lower bound of the set $\{b_n, b_{n+1}, \dots\}$ and hence we get
$$\inf{}_{j\ge n}(a_j) = \inf\{b_n, b_{n+1}, \dots\} \le \inf{}_{j\ge n}(b_j).$$
- This is a direct consequence of the two former properties. Let $u_n = \sup{}_{j=n}^{\infty}(a_j)$ and $v_n = \sup{}_{j=n}^{\infty}(b_j)$. Then we know that $u_n \le v_n$ for all $n \ge m$. Since both are again sequences of real numbers, we have
$$\inf{}_n(u_n) \le \inf{}_n(v_n).$$
But both quantities are precisely lim sup by definition. Therefore
$$\limsup_{n\to\infty} a_n = \inf{}_{n\ge m}\sup{}_{j\ge n}(a_j) \le \inf{}_{n\ge m}\sup{}_{j\ge n}(b_j) = \limsup_{n\to\infty} b_n.$$
- Again, this follows exactly the same line of reasoning as in the previous part.

□

Exercise 6.5. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that there exists a natural number M such that, for all $n \ge M$,
$$a_n \le b_n \le c_n.$$
Assume that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to the same limit L. Prove that $(b_n)_{n=m}^{\infty}$ converges to L. (Hint: use the previous exercise.)

Solution. We know from the previous exercise that
$$L = \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n \le \limsup_{n\to\infty} c_n = L.$$
Thus both limsup and liminf of $(b_n)_{n=m}^{\infty}$ coincide with L and therefore $(b_n)_{n=m}^{\infty}$ converges to L. □

Exercise 6.6. Let $x, y > 0$ be positive real numbers, and let $n, m \geq 1$ be positive integers. Prove:

(i) If $y = x^{1/n}$, then $y^n = x$.
(ii) If $y^n = x$, then $y = x^{1/n}$.
(iii) $x^{1/n}$ is a positive real number.
(iv) $x > y$ if and only if $x^{1/n} > y^{1/n}$.
(v) If $x > 1$ then $x^{1/n}$ decreases when n increases. If $x < 1$, then $x^{1/n}$ increases when n increases. If $x = 1$, then $x^{1/n} = 1$ for all positive integers n.
(vi) $(xy)^{1/n} = x^{1/n}y^{1/n}$.
(vii) $(x^{1/n})^{1/m} = x^{1/(nm)}$.

Remark. We remark that the exponentiation law (for integer exponents), which was originally proved for rational numbers, readily extends to that of real number even without modifying the argument. This is because the proof relies on something we call 'ordered field' structure (an algebraic system in which all the arithmetic operations work freely and the order relation is compatible with these operations in a usual way), and thankfully both the field $\mathbb{Q}$ of rationals and the field $\mathbb{R}$ of real numbers are ordered fields. *Solution.* With this remark, we will use the exponentiation laws in Proposition 4.27 and 4.29 (of the 1st lecture note) freely even in the context of real numbers instead of rationals.

Let $x, y > 0$ be positive real numbers, and let $m, n \geq 1$ be positive integers. Before the proof, we introduce the following notation

$$S(x) := \{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}, \quad x > 0.$$

Lemma 6.15 shows that $x^{1/n} := \sup S(x)$ is a non-negative real number.

(i) We prove $y^n = x$ for $y = x^{1/n}$ by contradiction. Assume that $y^n \neq x$. Then either $y^n < x$ or $y^n > x$.

- Assume that $y^n < x$. We claim that $(y + \varepsilon)^n < x$ for some $\varepsilon > 0$. Once this is proved, with any choice of such $\varepsilon > 0$, we get $y + \varepsilon \in S(x)$ and thus

$$y < y + \varepsilon \leq \sup S(x) = x^{1/n} = y,$$

a contradiction. Therefore the relation $y^n < x$ is impossible.
To complete the proof, we show the claim. Assuming otherwise, we get $(y+\varepsilon)^n > x$ for all $\varepsilon > 0$. In particular, by choosing $\varepsilon = \frac{1}{k}$, from Limit Laws we get

$$(y + \frac{1}{k})^n \geq x \implies y^n \geq x$$

as $k \to \infty$ a contradiction. This proves our claim and thus we are done.

- Assume that $y^n > x$. We claim that $(y - \varepsilon)^n < x$ for some $0 < \varepsilon < y$. Assuming this claim, we have

$$\forall z \in \mathbb{R}, \ z \geq y - \varepsilon \implies z^n \geq (y - \varepsilon)^n > x \implies z \notin S(x).$$

Taking contrapositive, whenever $z \in S(x)$ we must have $z < y - \varepsilon$. So $y - \varepsilon$ is an upper bound of $S(x)$ and this leads to the following contradiction:

$$y = \sup S(x) \leq y - \varepsilon < y.$$

Therefore $y^n > x$ is also impossible.

We complete the proof by proving our claim. If this is not the case, then for all $0 < \varepsilon < y$ we must have $(y-\varepsilon)^n \le x$. Now by the Archimedean property we choose N such that $Ny > 1$. Then we plug $\varepsilon = k^{-1}$ with $k > N$ (so that $0 < \varepsilon < y$) and take the limit as $k \to \infty$. Then we get

$$(y - k^{-1})^n \le x \implies y^n \le x,$$

as $k \to \infty$ contradicting our assumption.

(ii) Assume that $y^n = x$. Since $y > 0$ by the assumption, we have $y \in S(x)$. Now this shows $y \le x^{1/n}$. To prove the equality, assume otherwise. Then we get $y < x^{1/n}$. This means that y is not an upper bound of $S(x)$, so by negating the definition of upper bound we can pick some $z \in S(x)$ such that $y < z$. This gives $x = y^n < z^n \le x$, a contradiction! Therefore $y^n = x$ as desired.

(iii) We divide into two cases:

- First consider the case $0 < x \le 1$. By observing that $0 < x^n < x^{n-1} < \cdots \le x$ (or invoke the principle of mathematical induction to check this formally), we find that $x \in S(x)$. Thus we get

$$0 < x \le \sup S(x) = x^{1/n}.$$

- Now consider the case $x > 1$. Then $1^n = 1 < x$ and $1 \in S(x)$. So we get

$$1 \le \sup S(x) = x^{1/n}.$$

Therefore in any cases we get $x^{1/n} > 0$ and we are done.

(iv) ($\Rightarrow$): Assume that $x > y$. Then

$$\forall z \in \mathbb{R},\ z \in S(y) \implies z \ge 0 \text{ and } z^n \le y \implies z \ge 0 \text{ and } z^n \le x \implies z \in S(x)$$

and hence we obtain

$$x^{1/n} = \sup S(x) \ge \sup S(y) = y^{1/n}.$$

To exclude the possibility of having equality, assume otherwise so that $x^{1/n} = y^{1/n}$. Then the part (i) shows that

$$x = (x^{1/n})^n = (y^{1/n})^n = y,$$

which contradicts the assumption. Therefore we must have $x^{1/n} > y^{1/n}$.

($\Leftarrow$): From (i), we obtain

$$x = (x^{1/n})^n > (y^{1/n})^n = y.$$

(v) In view of (ii), it suffices to prove that $(x^{1/n}y^{1/n})^n = xy$. Indeed, by utilizing the exponentiation law $(ab)^n = a^n b^n$, we get

$$(x^{1/n}y^{1/n})^n = (x^{1/n})^n (y^{1/n})^n = xy$$

and therefore the assertion follows from (ii).

(vi) As in (v), it suffices to prove that $((x^{1/n})^{1/m})^{mn} = x$. By utilizing the exponentiation law $a^{mn} = (a^m)^n$, we get

$$((x^{1/n})^{1/m})^{mn} = (((x^{1/n})^{1/m})^m)^n = (x^{1/n})^n = x$$

and therefore the assertion follows from (i).

$\square$

Exercise 6.7. Let $x, y > 0$ be positive real numbers, and let q, r be rational numbers. Prove:

(i) x^q is a positive real number.
(ii) $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.
(iii) $x^{-q} = 1/x^q$.
(iv) If $q > 0$, then $x > y$ if and only if $x^q > y^q$.
(v) If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.

Solution.

(i) We have $x^{1/b} > 0$ by the previous exercise. Taking power to a, we still have a positive number. Therefore $x^q = (x^{1/b})^a > 0$.

(ii) Notice that $q + r = (ab' + a'b)/bb'$. Thus

$$\begin{aligned} x^{q+r} &= (x^{1/bb'})^{ab'+a'b} \\ &= (x^{1/bb'})^{ab'}(x^{1/bb'})^{a'b} \\ &= (x^{1/b})^a(x^{1/b'})^a \\ &= x^q x^r. \end{aligned}$$

To prove the second assertion, we first show that

$$(x^{1/b})^a = (x^a)^{1/b}. \qquad (1)$$

In view of (ii), it suffices to prove that $((x^{1/b})^a)^b = x^a$. But

$$((x^{1/b})^a)^b = (x^{1/b})^{ab} = ((x^{1/b})^b)^a = x^a$$

and hence (1) follows. Then by noting that $qr = aa'/bb'$, we have

$$\begin{aligned} x^{qr} &= (x^{1/bb'})^{aa'} \\ &= ((x^{1/b})^{1/b'})^{aa'} \\ &= (((x^{1/b})^{1/b'})^a)^{a'} \\ &= (((x^{1/b})^a)^{1/b'})^{a'} \\ &= (x^q)^r \end{aligned}$$

□

Exercise 6.8. Let $-1 < x < 1$. Show that $\lim_{n\to\infty} x^n = 0$. Using the identity $(1/x^n)x^n = 1$ for $x > 1$, conclude that x^n does not converge as $n \to \infty$ for $x > 1$.

Solution.

In view of the inequality

$$-|x|^n \le x^n \le |x|^n$$

and the Squeezing Theorem, it suffices to prove that $|x|^n \to 0$ as $n \to \infty$. Since $0 \le |x| < 1$, the sequence $a_n = |x|^n$ is non-increasing and bounded:

$$0 \le a_{n+1} \le a_n < 1.$$

So $(a_n)_1^\infty$ must converge to some real number L. But since

$$L = \lim_{n\to\infty} |x|^n = \lim_{n\to\infty} |x|^{n+1} = |x| \cdot L,$$

we have $(1 - |x|)L = 0$ and hence $L = 0$ as desired.

Now, in this case, assume that $x > 1$. Then $0 < x^{-1} < 1$ and hence we have $\lim_{n\to\infty} x^{-n} = 0$. But if it were true that $a := \lim_{n\to\infty} x^n$ exists, then by the Limit Laws,

$$a \cdot 0 = \lim_{n\to\infty} x^{-n} \cdot \lim_{n\to\infty} x^n = \lim_{n\to\infty} (x^{-n} x^n) = \lim_{n\to\infty} 1 = 1$$

a contradiction! Therefore $\lim_{n\to\infty} x^n$ cannot exist. □

7. Homework 7

Exercise 7.1. For any $x > 0$, show that $\lim_{n\to\infty} x^{1/n} = 1$. (Hint: first, given any $\varepsilon > 0$, show that $(1+\varepsilon)^n$ has no real upper bound M, as $n \to \infty$. To prove this claim, set $x = 1/(1+\varepsilon)$ and use Exercise 6.8. Now, with this preliminary claim, show that for any $\varepsilon > 0$ and for any real M, there exists a positive integer n such that $M^{1/n} < 1+\varepsilon$. Now, use these two claims, and consider the cases $x > 1$ and $x < 1$ separately.)

Solution.

(i) If $x = 1$, then the assertion is obvious since $1^{1/n} = 1$ yields the constant sequence with value 1.

(ii) Consider the case $x > 1$. We first prove that for any $\varepsilon > 0$, there exists N such that $x < (1+\varepsilon)^n$ whenever $n \geq N$. Indeed, from the previous exercise we have $\lim_{n\to\infty}(1+\varepsilon)^{-n} = 0$. Since $x^{-1} > 0$, there exists N such that

$$(1+\varepsilon)^{-n} = |(1+\varepsilon)^{-n} - 0| < x^{-1} \text{ whenever } n \geq N.$$

With this choice of N, we obtain the desired claim. Now for any $\varepsilon > 0$, let N be chosen as in the claim. Then whenever $n \geq N$, we have

$$1 < x \leq (1+\varepsilon)^n \implies 1 < x^{1/n} < 1+\varepsilon \implies |x^{1/n} - 1| < \varepsilon.$$

This proves that $x^{1/n}$ converges to 1 as $n \to \infty$.

(iii) Finally, consider the case $0 < x < 1$. Then $x^{-1} > 1$ and we know that $\lim_{n\to\infty}(x^{-1})^{1/n} = 1$. Therefore $x^{1/n} = ((x^{-1})^{1/n})^{-1}$ also converges to 1.

We also present an alternative solution of (ii):

(ii') If $x > 1$, then we know that $x^{1/n} > 1$. So if we write $\varepsilon_n = x^{1/n} - 1$, then we have $\varepsilon_n > 0$. Moreover,

$$x = (x^{1/n})^n = (1+\varepsilon_n)^n \geq 1 + n\varepsilon_n$$

and we get

$$0 \leq \varepsilon_n \leq \frac{x-1}{n}$$

By taking $n \to \infty$, Squeezing Theorem tells us that $\varepsilon_n \to 0$ as $n \to \infty$. This proves that $x^{1/n} = 1 + \varepsilon_n$ converges to 1. □

Exercise 7.2. Let $m \leq n < p$ be integers, let $(a_i)_{i=m}^n, (b_i)_{i=m}^n$ be a sequences of real numbers, let k be an integer, and let c be a real number. Prove:

- $$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i.$$
- $$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$

- $$\sum_{i=m}^{n}(a_i+b_i) = (\sum_{i=m}^{n} a_i) + (\sum_{i=m}^{n} b_i).$$
- $$\sum_{i=m}^{n}(ca_i) = c(\sum_{i=m}^{n} a_i).$$
- $$\left|\sum_{i=m}^{n} a_i\right| \leq \sum_{i=m}^{n} |a_i|.$$
- $$\text{If } a_i \leq b_i \text{ for all } m \leq i \leq n, \text{ then } \sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i.$$

Solution.

Before the proof, we remark that for any integer m and for any sequence $(a_i)_{i=m}^{\infty}$ of real numbers, we have

$$\sum_{i=m}^{m} a_i = a_m = 0 + a_m = a_m = \sum_{i=m}^{m-1} a_i + a_m.$$

This is also true for any sequence (a_i) of real numbers for which a_m is defined, since we can always restrict it to a smaller range.

- Let $m \leq n$ be integers. We prove the following statement by induction: **Claim.** *For any $l \in \mathbb{N}$ and for any sequence $(a_i)_{i=m}^{m+l}$ of real numbers, we have*

$$\sum_{i=m}^{n+l} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{n+l} a_i. \qquad (2) \tag{1}$$

Base case) First we consider the base case $l = 0$. In this case, Definition 7.1 shows that

$$\sum_{i=n+1}^{n+0} a_i = 0.$$

Thus this proves (2) as desired. **Induction step)** Next, we assume that the claim holds for l. Then since $n + l + 1 > n + 1$,

$$\begin{aligned}\sum_{i=m}^{n+l+1} a_i &= \left(\sum_{i=m}^{n+l} a_i\right) + a_{n+l+1} \\ &= \left(\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{n+l} a_i\right) + a_{n+l+1} \\ &= \sum_{i=m}^{n} a_i + \left(\sum_{i=n+1}^{n+l} a_i + a_{n+l+1}\right) \\ &= \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{n+l+1} a_i\end{aligned}$$

(by Definition 7.1) (by induction hypothesis) (by Definition 7.1) Therefore the claim follows by mathematical induction.

- Let m, k be integers. We prove the following statement by induction: **Claim.** *For any $l \in \mathbb{N}$ and for any sequence $(a_i)_{i=m}^{m+l}$ of real numbers, we have*

$$\sum_{i=m}^{m+l} a_i = \sum_{j=m+k}^{m+k+l} a_{j-k}. \qquad (3) \tag{2}$$

Base case) We consider the base case $l = 0$. Then both sides are given by

$$\sum_{i=m}^{m} a_i = a_m = a_{(m+k)-k} = \sum_{j=m+k}^{m+k} a_{j-k}$$

This proves the base case. **Induction step)** Assume that (3) holds for l. Then

$$\begin{aligned} \sum_{i=m}^{m+l+1} a_i &= \left(\sum_{i=m}^{m+l} a_i\right) + a_{m+l+1} \\ &= \left(\sum_{j=m+k}^{m+k+l} a_{j-k}\right) + a_{(m+k+l+1)-k} \\ &= \sum_{j=m+k}^{m+k+l+1} a_{j-k}. \end{aligned}$$

(by Definition 7.1) (by induction hypothesis)

Again, the claim follows from the principle of mathematical induction.

- Let m be an integer. **Claim.** *For any $l \in \mathbb{N}$ and for any sequences $(a_i)_{i=m}^{m+l}$ and $(b_i)_{i=m}^{m+l}$ of real numbers, we have*

$$\sum_{i=m}^{m+l}(a_i + b_i) = \sum_{i=m}^{m+l} a_i + \sum_{i=m}^{m+l} b_i. \qquad (4) \tag{3}$$

Base case) When $l = 0$, we have

$$\sum_{i=m}^{m}(a_i + b_i) = (a_m + b_m) = a_m + b_m = \sum_{i=m}^{m} a_i + \sum_{i=m}^{m} b_i.$$

Induction step) Suppose that (4) is true for l. Then

$$\begin{aligned} \sum_{i=m}^{m+l+1} (a_i + b_i) &= \left(\sum_{i=m}^{m+l}(a_i + b_i)\right) + (a_{m+l+1} + b_{m+l+1}) \\ &= \left(\sum_{i=m}^{m+l} a_i + \sum_{i=m}^{m+l} b_i\right) + (a_{m+l+1} + b_{m+l+1}) \\ &= \left(\sum_{i=m}^{m+l} a_i + a_{m+l+1}\right) + \left(\sum_{i=m}^{m+l} b_i + b_{m+l+1}\right) \\ &= \sum_{i=m}^{m+l+1} a_i + \sum_{i=m}^{m+l+1} b_i. \end{aligned}$$

- Let m be an integer and c be a real number. **Claim.** *For any $l \in \mathbb{N}$ and for any sequence $(a_i)_{i=m}^{m+l}$ of real numbers, we have*

$$\sum_{i=m}^{m+l}(ca_i) = c\left(\sum_{i=m}^{m+l} a_i\right). \qquad (5) \tag{4}$$

Base case) When $l = 0$, we have

$$\sum_{i=m}^{m}(ca_i) = ca_m = c\left(\sum_{i=m}^{m} a_i\right).$$

Induction step) Suppose that (5) is true for l. Then

$$\begin{aligned}\sum_{i=m}^{m+l+1}(ca_i) &= \left(\sum_{i=m}^{m+l}(ca_i)\right) + ca_{m+l+1} \\ &= c\left(\sum_{i=m}^{m+l} a_i\right) + ca_{m+l+1} \\ &= c\left(\sum_{i=m}^{m+l} a_i + a_{m+l+1}\right) \\ &= c\left(\sum_{i=m}^{m+l+1} a_i\right).\end{aligned}$$

This proves the claim as desired.

- Let m be an integer. **Claim.** *For any $l \in \mathbb{N}$ and for any sequence $(a_i)_{i=m}^{m+l}$ of real numbers, we have*

$$\left|\sum_{i=m}^{m+l} a_i\right| \leq \sum_{i=m}^{m+l} |a_i|. \qquad (6) \tag{5}$$

Base case) When $l = 0$, we have

$$\left|\sum_{i=m}^{m} a_i\right| = |a_m| = \sum_{i=m}^{m} |a_i|.$$

Induction step) Suppose that (6) is true for l. Then by the triangle inequality together with the induction hypothesis,

$$\begin{aligned}\left|\sum_{i=m}^{m+l+1} a_i\right| &= \left|\left(\sum_{i=m}^{m+l} a_i\right) + a_{m+l+1}\right| \\ &\leq \left|\sum_{i=m}^{m+l} a_i\right| + |a_{m+l+1}| \\ &\leq \sum_{i=m}^{m+l} |a_i| + |a_{m+l+1}| \\ &= \sum_{i=m}^{m+l+1} |a_i|.\end{aligned}$$

This proves the claim as desired.

- Let $m \leq n$ be integers, and let $(a_i)_{i=m}^{n}$ and $(b_i)_{i=m}^{n}$ be sequences of real numbers satisfying $a_i \leq b_i$ for $m \leq i \leq n$. What we want to prove is the following relation

$$\sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i. \qquad (7) \tag{6}$$

□

Exercise 7.3. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if: for every real number $\varepsilon > 0$, there exists an integer $N \geq M$ such that, for all $p, q \geq N$,

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon.$$

(Hint: recall that a sequence is convergent if and only if it is a Cauchy sequence.)

Solution.

The series $\sum_{n=m}^{\infty} a_n$ converges, by definition, exactly when the partial sum $S_N = \sum_{n=m}^{N} a_n$ converges as $N \to \infty$. Now using the completeness of $\mathbb{R}$, this happens exactly when $(S_N)_{N=m}^{\infty}$ is a Cauchy sequence. So it suffices to show that this is equivalent to the condition given in the exercise. This is almost trivial, but we introduce the proof anyway to please some meticulous readers.

- Suppose that the condition in the exercise holds. Let $\varepsilon > 0$ be arbitrary and let N be as in the condition. Then whenever $p, q \geq N$, either $p \geq q$ or $p \leq q$. In the former case,

$$|S_p - S_q| = \left|\sum_{n=q+1}^{p} a_n\right| < \varepsilon.$$

The latter case is treated exactly in the same way. Thus we obtain the following inequality unconditionally:

$$|S_p - S_q| < \varepsilon \quad \text{whenever } p, q \geq N.$$

This implies that $(S_N)_{N=m}^{\infty}$ is Cauchy as claimed.

- Suppose that $(S_N)_{N=m}^{\infty}$ is Cauchy. Then for any $\varepsilon > 0$, there exists N such that $|S_p - S_q| < \varepsilon$ whenever $p, q \geq N - 1$. (This $N - 1$ term is introduced for a very minute technical detail.)

  Now let $p, q > N$. Then we have either $q \geq p$ or $q < p$. In the former case, we get $\left|\sum_{n=p}^{q} a_n\right| = |S_{q-1} - S_{p-1}| < \varepsilon$. In the latter case, $\left|\sum_{n=p}^{q} a_n\right| = 0 < \varepsilon$ by definition of summation notation. This implies the condition in the exercise.

Therefore the equivalence is proved. □

Exercise 7.4. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge. (Hint: use Exercise 12.20.)

Solution. Assume that $\sum_{n=m}^{\infty} a_n$ converges. Using the previous exercise, for arbitrary $\varepsilon > 0$, there exists N such that

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon \quad \text{for any } p, q \geq N.$$

Now let $n \geq N$. Then by choosing p, q by $p = q = n$, we get

$$|a_n - 0| = |a_n| = \left|\sum_{k=n}^{n} a_k\right| < \varepsilon.$$

Therefore $(a_n)_{n=m}^{\infty}$ converges to 0 as desired. □

Exercise 7.5. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is convergent. Moreover,

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Solution. Assume that $\sum_{n=m}^{\infty} a_n$ converges absolutely. Then by Exercise 1, for any $\varepsilon > 0$ there exists N such that

$$\left|\sum_{n=p}^{q} |a_n|\right| < \varepsilon \quad \text{whenever } p, q \geq N.$$

Now by the property of the summation, we get

$$\left|\sum_{n=p}^{q} a_n\right| \leq \sum_{n=p}^{q} |a_n| < \varepsilon \quad \text{whenever } p, q \geq N.$$

Thus by Exercise 1 again, the series $\sum_{n=m}^{\infty} a_n$ converges as well. Moreover, since the partial sum $T_N = \sum_{n=m}^{N} |a_n|$ is monotone increasing,

$$\left|\sum_{n=m}^{N} a_n\right| \leq \sum_{n=m}^{N} |a_n| = T_N \leq \sup T_N = \lim_{N\to\infty} T_N = \sum_{n=m}^{\infty} |a_n|.$$

This essentially proves the desired inequality. To argue rigorously, let $A = \sum_{n=m}^{\infty} |a_n|$. Then the above inequality tells us that

$$\left| \sum_{n=m}^{N} a_n \right| \leq A \text{ for all } N \geq m \implies -A \leq \sum_{n=m}^{\infty} a_n \leq A \implies \left| \sum_{n=m}^{\infty} a_n \right| \leq A.$$

This completes the proof. □

Exercise 7.6.

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let $\sum_{n=m}^{\infty} b_n$ be a series of real numbers converging to y. Then $\sum_{n=m}^{\infty}(a_n + b_n)$ is a convergent series that converges to $x + y$. That is,

$$\sum_{n=m}^{\infty}(a_n + b_n) = (\sum_{n=m}^{\infty} a_n) + (\sum_{n=m}^{\infty} b_n).$$

(Deferred)

Exercise 7.7. Let $\sum_{n=m}^{\infty} a_n, \sum_{n=m}^{\infty} b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Moreover,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Solution. The proof is essentially an imitation of that of Exercise 3: Assume that $\sum_{n=m}^{\infty} b_n$ converges. By Exercise 1, for any $\varepsilon > 0$ there exists N such that

$$\left| \sum_{n=p}^{q} b_n \right| < \varepsilon \quad \text{whenever } p, q \geq N.$$

But since

$$\left| \sum_{n=p}^{q} a_n \right| \leq \sum_{n=p}^{q} |a_n| \leq \sum_{n=p}^{q} |b_n| < \varepsilon,$$

it follows from Exercise 1 that $\sum_{n=m}^{\infty} |a_n|$ converges as well. Now let $T_N = \sum_{n=m}^{N} |b_n|$ be the partial sum of $(|b_n|)_{n=m}^{\infty}$. Then $(T_N)_{N=m}^{\infty}$ is monotone increasing and convergent. So we have

$$\left| \sum_{n=m}^{N} a_n \right| \leq \sum_{n=m}^{N} |a_n| \leq T_N \leq \sup T_N = \lim_{N\to\infty} T_N = \sum_{n=m}^{\infty} |b_n|.$$

Thus it follows that

$$\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} |b_n|.$$

Combining this with Exercise 3, it follows that $\sum_{n=m}^{\infty} a_n$ converges absolutely and

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} |b_n|.$$

□

Exercise 7.8. For any $n \in \mathbb{N}$, define $a_n := (-1)^{n+1}/(n+1)$. Find a bijection $g\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=0}^{\infty} a_{g(n)}$ diverges.

Solution. We described the solution in class. Here is a sketch of the argument. It is easiest to describe what to do in words. First, sum up the odd terms only, such that the partial sum up to a certain index N is at least 10. This is possible to do since $\sum_{k=1}^{\infty} 1/(2k)$ diverges. Then sum the first negative term ($n = 0$). Then, sum the next several odd terms so the partial sum up to another index M is at least 20. Once again, this is possible to do since $\sum_{k=1}^{\infty} 1/(2k)$ diverges. Then sum the next negative term ($n = 2$). Then, sum the next several odd terms so the partial sum up to another index P is at least 30. Once again, this is possible to do since $\sum_{k=1}^{\infty} 1/(2k)$ diverges. Then sum the next negative term ($n = 4$). Repeat this process. By construction, the partial sums do not converge, since they increase without bound. Rearranging terms in the sum corresponds to defining a bijection g as specified.

$\square$

Exercise 7.9. Let $(b_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n\to\infty} b_n^{1/n}.$$

Solution. Let $L := \liminf_{n\to\infty} \frac{b_{n+1}}{b_n}$. If $L = 0$ there is nothing to show, so we assume that $L > 0$. Assume for now that $L < \infty$.

Let $\varepsilon > 0$ such that $L - \varepsilon > 0$. From the Proposition characterizing liminf, there exists an integer $N \geq m$ such that, for all $n \geq N$, we have $(b_{n+1}/b_n) \geq L - \varepsilon$. That is, $b_{n+1} \geq (L-\varepsilon)b_n$. By induction, we conclude that, for all $n \geq N$,

$$b_n \geq (L-\varepsilon)^{n-N} b_N.$$

That is, for all $n \geq N$,

$$b_n^{1/n} \geq (b_N(L-\varepsilon)^{-N})^{1/n}(L+\varepsilon). \qquad (*)$$

Letting $n \to \infty$ on the right side of $(*)$, and applying the Limit Laws and a Lemma from the notes stating that $\lim_{n\to\infty} z^{1/n} = 1$ where $z = L - \varepsilon > 0$,

$$\lim_{n\to\infty} (b_N(L-\varepsilon)^{-N})^{1/n}(L-\varepsilon) = L - \varepsilon.$$

So, applying the Comparison Principle to $(*)$,

$$\liminf_{n\to\infty} b_n^{1/n} \geq L - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\liminf_{n\to\infty} b_n^{1/n} \geq L$, as desired, in the case $L < \infty$. In the case $L = \infty$, we note that the above argument shows that $\liminf_{n\to\infty} b_n^{1/n} \geq L'$ for any $L' > 0$. $\square$

Exercise 7.10. Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Also, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and if $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

Exercise 7.11. Give an example of two convergent series of real numbers $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ such that the series $\sum_{n=0}^{\infty}(a_n b_n)$ is not convergent.

Solution. Let $a_n = b_n = 1/\sqrt{n}$ for any $n \geq 1$ (and let $a_0 = b_0 = 0$). Then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge by the dyadic criterion (or Corollary 2.6.33) but $\sum_{n=0}^{\infty}(a_n b_n) = \sum_{n=0}^{\infty} 1/n$ diverges , again from the dyadic criterion (or Corollary 2.6.33). □

Exercise 7.12. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number.

- If the sequence $(a_n)_{n=0}^{\infty}$ converges to L, then every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.
- Conversely, if every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L, then $(a_n)_{n=0}^{\infty}$ itself converges to L.

8. Homework 8

Preliminary. In this section we deal with some facts that are relevant to our problems but can be coped with only previous materials.

Maximum and Minimum of subsets of $\mathbb{R}$. Let E be a non-empty subset of $\mathbb{R}$. If there is an element $M \in E$ such that $x \leq M$ for any element $x \in E$, we call M the maximum of E and denote $M = \max E$. Similarly, if there is an element $m \in E$ such that $x \geq m$ for any element $x \in E$, we call m the minimum of E and denote $m = \min E$.

This concept is quite close to that of supremum and infimum, but the difference is that maximum and minimum need not always exist.

Proposition 1.1. *Let $E \subseteq \mathbb{R}$. Then*

- *E have at most one maximum and at most one minimum.*
- *If $\max E$ exists, then $\max E = \sup E$.*
- *If $\min E$ exists, then $\min E = \inf E$.*

Remark. The first assertion justifies our notation as well as our usage of the definite article 'the'.

Proof. Suppose that M, M' are maximums of E. Since $M, M' \in E$, we must have $M' \leq M$ and $M \leq M'$. This implies $M = M'$ and hence there cannot exist two or more maximums. The same argument applies for the uniqueness of minimum.

Now assume that $M = \max E$ exists. Then M is an upper bound of E. Moreover, M is also the least upper bound since any $x < M$ cannot be an upper bound of E. Therefore $M = \sup E$.

The third assertion follows in exactly the same manner.

Exercise 8.1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers converging to 0. Show that $(|a_n|)_{n=m}^{\infty}$ also converges to zero.

Solution. By definition, for any $\varepsilon > 0$ there exists $N > m$ such that whenever $n \geq N$ we have $|a_n - 0| \leq \varepsilon$. But since $|a_n - 0| = |a_n| = ||a_n| - 0|$, we also have $||a_n| - 0| < \varepsilon$. By reading out this result using definition again, we have $\lim_{n\to\infty} |a_n| = 0$ as desired. □

Exercise 8.2. Let $a < b$ be real numbers. Let I be any of the four intervals (a, b), $(a, b]$, $[a, b)$ or $[a, b]$. Then the closure of I is $[a, b]$.

Solution. Let $\overline{I}$ denote the closure of I. We prove $\overline{I} = [a, b]$ by showing that a real number x lies in $\overline{I}$ exactly when $x \in [a, b]$. To this end, we divide the case according to whether x lies inside I or not.

- Suppose that $x \in (a, b)$. Then $x \in I$ and x is an adherent point of I. Thus $x \in \overline{I}$.

- Suppose that x is either a or b. Let us first consider the case where $x = a$. Then for any $\varepsilon > 0$, there exists y such that $a < y < \min\{a+\varepsilon, b\}$. Then $y \in I$ and $|a-y| < \varepsilon$. This proves that $a \in \overline{I}$. The proof of $b \in \overline{I}$ is quite the same.
- Suppose that $x \notin [a, b]$. That is, either $x < a$ or $x > b$. Let us examine the case $x < a$ first. Then for $\varepsilon = a - x$, we find that there is no $y \in I$ satisfying $|x - y| < \varepsilon$. Indeed, for any $y \in I$ we have $y \geq a > x$ and

  $$|y - x| = y - x \geq a - x = \varepsilon.$$

  This shows that $x \notin \overline{I}$. The case $x > b$ can be treated in a similar way, proving that $x \notin \overline{I}$.

Therefore $\overline{I} = [a, b]$. □

Exercise 8.3. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are equivalent. (That is, one statement is true if and only if the other statement is true.)

(i) f converges to L at x_0 in E.
(ii) For every sequence $(a_n)_{n=0}^{\infty}$ in E, and which converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L.

Solution. (i) $\Rightarrow$ (ii) : Let $(a_n)_{n=0}^{\infty}$ be any sequence in E that converges to x_0. To prove that $\lim_{n\to\infty} f(a_n) = L$, let $\varepsilon > 0$ be arbitrary. Then

- Using Definition 2.14, pick $\delta = \delta(\varepsilon) > 0$ such that for any $x \in E$ with $|x - x_0| < \delta$ we have $|f(x) - L| < \varepsilon$.
- Using the definition of convergence of sequence, pick $N = N(\delta)$ such that for any $n \geq N$ we have $|a_n - x_0| < \delta$.

Combining these two facts, we find that $|f(a_n) - L| < \varepsilon$ holds whenever $n \geq N$. Therefore $(f(a_n))_{n=0}^{\infty}$ converges to L.

(ii) $\Rightarrow$ (i) : We prove the contrapositive. Assume that f does not converge to L as $x \to x_0$ in E. By negating Definition 2.14, we find that

- There exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $x \in E$ such that $|x - x_0| < \delta$ but $|f(x) - L| \geq \varepsilon$.

Now for each particular choice $\delta = n^{-1}$ (where $n \in \mathbb{N}$), we utilize this statement to pick some $x = a_n \in E$ such that $|a_n - x_0| < \delta = n^{-1}$ but $|f(a_n) - L| \geq \varepsilon$.

On the one hand, by this construction we clearly have $\lim_{n\to\infty} a_n = x_0$. (Just apply the squeezing theorem to $x_0 - n^{-1} < a_n < x_0 + n^{-1}$.) On the other hand, $(f(a_n))_{n=0}^{\infty}$ cannot converge to L. Indeed, assume otherwise so that $f(a_n)$ converges to L. Then there exists N such that whenever $n > N$ we have $|f(a_n) - L| < \frac{1}{2}\varepsilon$. But since $|f(a_n) - L| \geq \varepsilon$ always holds, we must have $0 < \varepsilon < \frac{1}{2}\varepsilon$, a contradiction! This completes the proof. □

Exercise 8.4. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, let L be a real number, and let δ be a positive real number. Then the following two statements are equivalent:

(i) $\lim_{x\to x_0; x\in E} f(x) = L$.
(ii) $\lim_{x\to x_0; x\in E\cap(x_0-\delta, x_0+\delta)} f(x) = L$.

Solution. For the simplicity of notation, let us denote $E_\delta = E \cap (x_0 - \delta, x_0 + \delta)$.

(i) $\Rightarrow$ (ii) : This direction is almost trivial. Assume that $\lim_{x\to x_0;x\in E} f(x) = L$. Let $\varepsilon > 0$ be arbitrary. Then there exists $\eta > 0$ such that, for any $x \in E$ with $|x - x_0| < \eta$ we have $|f(x) - L| < \varepsilon$. So if $x \in E_\delta$ and $|x - x_0| < \eta$, then we have $x \in E$ and hence $|f(x) - L| < \varepsilon$. From this we read out that $\lim_{x\to x_0;x\in E_\delta} f(x) = L$.

(ii) $\Rightarrow$ (i) : Assume that $\lim_{x\to x_0;x\in E_\delta} f(x) = L$. That is, for any $\varepsilon > 0$, there exists $\eta > 0$ such that whenever $x \in E_\delta$ and $|x - x_0| < \eta$, we have $|f(x) - L| < \varepsilon$. To complete the proof, let $\eta' = \min\{\eta, \delta\}$. Then whenever $x \in E$ and $|x - x_0| < \eta'$, we have both $x \in E_\delta$ and $|x - x_0| < \eta$. Then $|f(x) - L| < \varepsilon$. From this we read out that $\lim_{x\to x_0;x\in E} f(x) = L$.

□

Exercise 8.5. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in X$. Then the following three statements are equivalent.

(i) f is continuous at x_0.

(ii) For every sequence $(a_n)_{n=0}^{\infty}$ in X such that $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.

(iii) For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x \in X$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.

Solution. (i) $\Leftrightarrow$ (ii) : f is continuous at x_0 if and only if $\lim_{x\to x_0;x\in X} f(x) = f(x_0)$. By Exercise 3, this is true if and only if (ii) is true.

(i) $\Leftrightarrow$ (iii) : The statement (iii), together with the choice $E = X$ in Definition 2.14, exactly tells us that $\lim_{x\to x_0;x\in X} f(x) = f(x_0)$, which is the definition of the continuity of f at x_0.

□

Exercise 8.6. Let X, Y be subsets of $\mathbb{R}$. Let $f : X \to Y$ and let $g : Y \to \mathbb{R}$ be functions. Let $x_0 \in X$. If f is continuous at x_0, and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Solution. We have $\lim_{x\to x_0;x\in X} f(x) = f(x_0)$ and $\lim_{y\to f(x_0);y\in Y} g(y) = g(f(x_0))$. Let $\varepsilon > 0$ be arbitrary. Using Definition 2.14,

- We can pick $\eta = \eta(\varepsilon) > 0$ such that whenever $y \in Y$ and $|y - f(x_0)| < \eta$ we have $|g(y) - g(f(x_0))| < \varepsilon$.
- We can pick $\delta = \delta(\eta) > 0$ such that whenever $x \in X$ and $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \eta$.

Combining these two statements, we find that whenever $|x - x_0| < \delta$, we have $f(x) \in Y$ and $|f(x) - f(x_0)| < \eta$, hence $|g(f(x)) - g(f(x_0))| < \varepsilon$. From this we read out that $\lim_{x\to x_0;x\in X} g(f(x)) = g(f(x_0))$. □

Exercise 8.7. Let $a < b$ be real numbers. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x\in[a,b]} f(x)$ be the maximum value of f on $[a, b]$, and let $m := \inf_{x\in[a,b]} f(x)$ be the minimum value of f on $[a, b]$. Let y be a real number such that $m \le y \le M$. Then there exists $c \in [a, b]$ such that $f(c) = y$. Moreover, $f([a, b]) = [m, M]$.

Solution. By the Maximum Principle, both the values M and m are achieved at some different points in $[a, b]$. That is, there exists $a' < b'$ in $[a, b]$ such that $\{m, M\} = \{f(a'), f(b')\}$.[2] Now

[2]This tricky demonstration is a technical, brief way of saying that 'one of a' and b' is a maximum point and the other is a minimum point'.

note that f is continuous on $[a', b']$ as well. Thus if $m \le y \le M$, then by the Intermediate Value Theorem, there exists $c \in [a', b']$ such that $f(c) = y$. Since $c \in [a, b]$ as well, the first assertion follows.

For the second assertion, we prove that $[m, M] \subseteq f([a, b])$ and $f([a, b]) \subseteq [m, M]$ are subsets of each other. To this end, just observe that

- For any $y \in [m, M]$ we have $f(c) = y$ for some $c \in [a, b]$ by the first assertion. So we have $y \in f([a, b])$.
- If $y \in f([a, b])$, then $y = f(c)$ for some $c \in [a, b]$. Then

$$m = \inf{}_{x \in [a,b]} f(x) \le f(c) \le \sup{}_{x \in [a,b]} f(x) = M$$

  and hence $y \in [m, M]$.

This shows that $[m, M] \subseteq f([a, b])$ and $f([a, b]) \subseteq [m, M]$. Therefore they are equal to each other. □

Exercise 8.8. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be two sequences of real numbers. Then $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$.

Solution. Carefully read out the definition of equivalent sequences to convince yourself that the observation $|a_n - b_n| = ||a_n - b_n| - 0|$ suffices to complete the proof. □

Exercise 8.9. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function. Assume that there exists a real number $L > 0$ such that, for all $x, y \in [a, b]$, we have $|f(x)-f(y)| \le L|x-y|$. Such an f is called Lipschitz continuous. Prove that f is continuous. Then, find a continuous function that is not Lipschitz continuous.

Solution. For any $\varepsilon > 0$, pick $\delta = \varepsilon/(L + 1)$. Then whenever $x, y \in [a, b]$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| \le L|x - y| \le L \cdot \frac{\varepsilon}{L + 1} < \varepsilon.$$

This proves that f is continuous at any point. (And also proves that f is uniformly continuous.)

An example of a function which is continuous but not Lipschitz continuous is $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. To check this, notice that for any $n \in \mathbb{N}$,

$$|f(1/n) - f(0)| = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n}} = \frac{\sqrt{n}}{n} = \sqrt{\frac{1}{n}} = \sqrt{|\frac{1}{n} - 0|}.$$

Thus no number $L \ge 0$ cannot satisfy $|f(x)-f(y)| \le L|x-y|$ for all $x, y \in [0, 1]$. (Otherwise we must be able to find some $L \ge 0$ satisfying $L \ge \sqrt{n}$ for all n, which is impossible.) □

Exercise 8.10. Let X be a subset of $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Then the following two statements are equivalent.

(i) f is uniformly continuous on X.

(ii) For any two equivalent sequences $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ in X, the sequences $(f(a_n))_{n=m}^{\infty}$, $(f(b_n))_{n=m}^{\infty}$ are also equivalent sequences.

Solution. (i) ⇒ (ii): Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences in X which are equivalent. Then for any $\varepsilon > 0$,

- By the uniform convergence of f, there exists $\delta > 0$ such that whenever $x, y \in X$ and $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

- By the equivalence, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ we have $|a_n - b_n| < \delta$.

Combining two statements, it follows that whenever $n \geq N$ we have $|f(a_n) - f(b_n)| < \varepsilon$. This proves that $(f(a_n))_{n=m}^{\infty}$, $(f(b_n))_{n=m}^{\infty}$ are also equivalent sequences.

(ii) $\Rightarrow$ (i): We prove contrapositive. Let us assume that f is not uniformly continuous. By negating Definition 3.31,

- There exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $x, y \in X$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Now for each particular choice $\delta = n^{-1}$ (where $n \in \{1, 2, \dots\}$) pick such two elements $x = a_n, y = b_n \in X$ (that is, $|a_n - b_n| < \delta = n^{-1}$ but $|f(a_n) - f(b_n)| \geq \varepsilon$). Then $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are equivalent but $(f(a_n))_{n=m}^{\infty}$, $(f(b_n))_{n=m}^{\infty}$ cannot be equivalent. This proves the contrapositive as desired. □

Exercise 8.11. Give an example of a continuous function $f : \mathbb{R} \to (0, \infty)$ such that, for any real number $0 < \varepsilon < 1$, there exists $x \in \mathbb{R}$ such that $f(x) = \varepsilon$.

Remark. In view of the Maximum Principle, a continuous function $f : \mathbb{R} \to (0, \infty)$ must attain positive minimum on any finite closed interval $[a, b] \subseteq \mathbb{R}$. Consequently, if $\varepsilon > 0$ is small, any solution of $f(x) = \varepsilon$ have large size. This in particular suggests that any continuous function $f : \mathbb{R} \to (0, \infty)$ satisfying $\lim_{x\to\infty} f(x) = 0$ serves an example.

- *1st Solution.* Let $f(x) = 2^{-x}$. Then clearly f is a continuous function with range $(0, \infty)$. Moreover, for any $0 < \varepsilon < 1$ we have $f(\log_2 \frac{1}{\varepsilon}) = \varepsilon$. Therefore f satisfies all the desired properties.

If you raise an objection by claiming that we have never learned both exponential function and logarithm in this course, here is another solution:

- *2nd Solution.* Let $f : \mathbb{R} \to (0, \infty)$ by
$$f(x) = \frac{1}{1 + x^2}.$$

9. Homework 9

Exercise 9.1. Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbb{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

Solution. We have
$$\lim_{x\to x_0; x\in X\setminus\{x_0\}} (f(x) - f(x_0)) = \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0) \cdot 0 = 0.$$
Therefore, by adding $f(x_0)$ to both sides, it follows that f is continuous at x_0. □

Exercise 9.2. Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, let $f : X \to \mathbb{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.

(i) f is differentiable at x_0 on X with derivative L.
(ii) For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then
$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon |x - x_0|.$$

Solution. The proof is almost a tautology. Nevertheless we spell out every detail.

(i) $\Rightarrow$ (ii) : By definition, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X \backslash \{x_0\}$ satisfies $|x - x_0| < \delta$, then

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| < \varepsilon.$$

Now multiply both sides by $|x - x_0|$. Then we have

$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon|x - x_0|.$$

Since this continues to hold when $x = x_0$, we get (ii).

(ii) $\Rightarrow$ (i) : For every $\varepsilon > 0$, pick $\delta = \delta(\varepsilon/2) > 0$ as in (ii). Then whenever $x \in X \backslash \{x_0\}$ and $|x - x_0| < \delta$, we have

$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \frac{\varepsilon}{2}|x - x_0|$$

$$\implies \left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

□

Exercise 9.3. Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions.

(i) If f is constant, so that there exists $c \in \mathbb{R}$ such that $f(x) = c$, then f is differentiable at x_0 and $f'(x_0) = 0$.

(ii) If f is the identity function, so that $f(x) = x$, then f is differentiable at x_0 and $f'(x_0) = 1$.

(iii) If f, g are differentiable at x_0, then $f + g$ is differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$. (**Sum Rule**)

(iv) If f, g are differentiable at x_0, then fg is differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$. (**Product Rule**)

(v) If f is differentiable at x_0, and if $c \in \mathbb{R}$, then cf is differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.

(vi) If f, g are differentiable at x_0, then $f - g$ is differentiable at x_0, and $(f - g)'(x_0) = f'(x_0) - g'(x_0)$.

(vii) If g is differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then $1/g$ is differentiable at x_0, and $(1/g)'(x_0) = -g'(x_0)/(g(x_0))^2$.

(viii) If f, g are differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable at x_0, and

$$(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

(**Quotient Rule**)

Solution.

(i) Notice that

$$\frac{f(x) - f(x_0)}{x - x_0} = 0, \quad x \in X \backslash \{x_0\}.$$

Taking limit as $x \to x_0$ for $x \in X \backslash \{x_0\}$, it follows that f is differentiable at x_0 and $f'(x_0) = 0$.

(ii) Notice that
$$\frac{f(x)-f(x_0)}{x-x_0}=1, \quad x\in X\backslash\{x_0\}.$$
Taking limit as $x\to x_0$ for $x\in X\backslash\{x_0\}$, it follows that f is differentiable at x_0 and $f'(x_0)=1$.

(iii) Notice that
$$\frac{[f(x)+g(x)]-[f(x_0)+g(x_0)]}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}, \quad x\in X\backslash\{x_0\}.$$
Taking limit as $x\to x_0$ for $x\in X\backslash\{x_0\}$, we know from both the assumption and the Limit Laws that the right-hand side converges to $f'(x_0)+g'(x_0)$. This proves the Sum Rule.

(iv) Notice that
$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}\cdot g(x_0)+\frac{g(x)-g(x_0)}{x-x_0}\cdot f(x), \quad x\in X\backslash\{x_0\}.$$
Taking limit as $x\to x_0$ for $x\in X\backslash\{x_0\}$, we know from both the assumption and the Limit Laws that the right-hand side converges to $f'(x_0)g(x_0)+g'(x_0)f(x_0)$. This proves the Product Rule.

(v) This follows by (i) and (iv).

(vi) This follows by (iii) and (v), with the choice $c=-1$.

(vii) Let ϕ_g be as in (1.1) for g. Then
$$\frac{[1/g(x)]-[1/g(x_0)]}{x-x_0}=-\frac{g(x)-g(x_0)}{x-x_0}\frac{1}{g(x)g(x_0)}, \quad x\in X\backslash\{x_0\}.$$
Taking limit as $x\to x_0$ for $x\in X\backslash\{x_0\}$, we know from both the assumption and the Limit Laws that the right-hand side converges to $-g'(x_0)/(g(x_0))^2$. This proves (vii).

(viii) This follows from (iv) and (vii).

□

Exercise 9.4. Let X,Y be subsets of $\mathbb{R}$, let $x_0\in X$ be a limit point of X, and let $y_0\in Y$ be a limit point of Y. Let $f:X\to Y$ be a function such that $f(x_0)=y_0$ and such that f is differentiable at x_0. Let $g:Y\to\mathbb{R}$ be a function that is differentiable at y_0. Then the function $g\circ f:X\to\mathbb{R}$ is differentiable at x_0, and
$$(g\circ f)'(x_0)=g'(y_0)f'(x_0).$$

Remark. This would have followed easily if it were true that
$$\frac{g(f(x))-g(f(x_0))}{x-x_0}=\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}\cdot\frac{f(x)-f(x_0)}{x-x_0}.$$
Unfortunately, this is not always true as $f(x)-f(x_0)$ may vanish infinitely many times near x_0. We need to circumvent this technical issue.

Let $(a_n)_{n=0}^{\infty}$ be any sequence in $X\setminus\{x_0\}$ which converges to x_0. Then we deduce from Exercise 8.5 that $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$. Now we divide into two cases:

- *Case 1)* Assume that $f'(x_0)\neq 0$. Then for the choice $\varepsilon=\frac{1}{2}|f'(x_0)|>0$, there exists $\delta>0$ such that, whenever $x\in X$ and $|x-x_0|<\delta$ we have
$$|f(x)-f(x_0)-f'(x_0)(x-x_0)|\leq\varepsilon|x-x_0|.$$

Then it follows from the reverse triangle inequality that

$$\begin{aligned}|f(x) - f(x_0)| &\geq |f'(x_0)(x - x_0)| - |f(x) - f(x_0) - f'(x_0)(x - x_0)| \\ &\geq |f'(x_0)||x - x_0| - \frac{1}{2}|f'(x_0)||x - x_0| = \frac{1}{2}|f'(x_0)||x - x_0|.\end{aligned}$$

since $|f'(x_0)| = 2\varepsilon$. In particular, since $|x-x_0| \neq 0$, this implies that $|f(x)-f(x_0)| > 0$ whenever $|x-x_0| < \delta$. Next, we pick N sufficiently large so that $|a_n-x_0| < \delta$ whenever $n \geq N$. Since $a_n \in X\backslash\{x_0\}$, it follows that $|f(a_n) - f(x_0)| > 0$. Then our intuitive idea works and we have

$$\begin{aligned}\lim_{n\to\infty} \frac{g(f(a_n)) - g(f(x_0))}{a_n - x_0} &= \lim_{n\to\infty} \frac{g(f(a_n)) - g(f(x_0))}{f(a_n) - f(x_0)} \frac{f(a_n) - f(x_0)}{a_n - x_0} \\ &= g'(y_0)f'(x_0).\end{aligned}$$

- *Case 2)* Now assume that $f'(x_0) = 0$. For $\varepsilon = \frac{\sqrt{2}}{4}$, pick $\delta > 0$ such that, if $y \in Y$ and $|y - y_0| < \delta$ then

$$|g(y) - g(y_0) - g'(y_0)(y - y_0)| \leq \frac{\varepsilon}{2}|y - y_0|.$$

Then by the triangle inequality, we have

$$\begin{aligned}|g(y) - g(y_0)| &\leq |g'(y_0)||y - y_0| + |g(y) - g(y_0) - g'(y_0)(y - y_0)| \\ &\leq (|g'(y_0)| + \frac{\varepsilon}{2})|y - y_0|.\end{aligned}$$

Also, choose N sufficiently large so that whenever $n \geq N$, we have $|f(a_n)-f(x_0)| < \delta$. Then by noting that

$$\left|\frac{g(f(a_n)) - g(f(x_0))}{a_n - x_0}\right| \leq \left(|g'(y_0)| + \frac{\varepsilon}{2}\right) \frac{|f(a_n) - f(x_0)|}{|a_n - x_0|}$$

it follows from the squeezing theorem, together with $\lim_{n\to\infty} \frac{f(a_n)-f(x_0)}{a_n-x_0} = f'(x_0) = 0$, we have

$$\lim_{n\to\infty} \frac{g(f(a_n)) - g(f(x_0))}{a_n - x_0} = 0 = g'(y_0)f'(x_0).$$

Therefore, in any cases the Chain Rule follows.

Exercise 9.5. Let $a < b$ be real numbers, and let $f : (a, b) \to \mathbb{R}$ be a function. If $x_0 \in (a, b)$, if f is differentiable at x_0, and if f attains a local maximum or minimum at x_0, then $f'(x_0) = 0$.

Solution. We first consider local maximum case. Since x_0 is a local maximum point of f, there exists a sufficiently small $\delta > 0$ such that f attains a maximum on $(x_0 - \delta, x_0 + \delta) \subseteq (a, b)$. Then we have

$$\frac{f(x) - f(x_0)}{x - x_0} \begin{cases} \geq 0 & x_0 - \delta < x < x_0 \\ \leq 0 & x_0 < x < x_0 + \delta. \end{cases}$$

Thus taking right-limit and left-limit, we find that

$$f'(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{and} \quad f'(x_0) = \lim_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

This proves that $f'(x_0) = 0$. Local minimum case can be tackled in the same way. □

Exercise 9.6. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Assume that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

Solution. Utilizing the Maximum Principle, pick two points $x_0, x_1 \in [a, b]$ such that f attains a global maximum at x_0 and a global minimum at x_1. If $f(x_0) = f(x_1)$, then f must reduce to a constant function and hence the claim follows from Exercise 8.7.(i). (In this case, you can pick any point $x \in (a, b)$.) In view of the previous observation, we may ssume $f(x_0) \neq f(x_1)$. This implies that either $f(x_0) \neq f(a)$ or $f(x_1) \neq f(a)$. In either cases, there exists a global extremum x of f, which is neither a nor b. Thus x is also a global extremum of $f|_{(a,b)}$. Then x is a local extremum of $f|_{(a,b)}$ and by the previous exercise, we have $f'(x) = 0$. □

Exercise 9.7. Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbb{R}$ be a function.

- If f is monotone increasing and if f is differentiable at x_0, then $f'(x_0) \geq 0$.
- If f is monotone decreasing and if f is differentiable at x_0, then $f'(x_0) \leq 0$.

Solution. We first assume that f is monotone. If $x \in X \backslash \{x_0\}$, then by dividing the cases based on whether $x > x_0$ or $x < x_0$, we find that

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

always holds. Now assume further that f is differentiable at x_0. Taking $x \to x_0$ in $X \backslash \{x_0\}$, the inequality is preserved and hence we have

$$f'(x_0) = \lim_{x \to x_0; x \in X \backslash \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

For ...

□

Exercise 9.8. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.

10. Homework 10

10.1. **Properties of supremum infimum combined with arithmetic operations.**

Lemma 1. *Let $A \subseteq \mathbb{R}$ be a non-empty subset and $c \in \mathbb{R}$. Define $c + A = \{c + a : a \in A\}$. Then* $\sup(c + A) = c + (\sup A)$ *and* $\inf(c + A) = c + (\inf A)$.

Lemma 2. *Let $E \subseteq \mathbb{R}$ be a non-empty subset and $c \in \mathbb{R}$. Define $cA = \{ca : a \in A\}$. Then*

- *If $c > 0$, then* $\sup(cA) = c(\sup A)$ *and* $\inf(cA) = c(\inf A)$.
- *If $c < 0$, then* $\sup(cA) = c(\inf A)$ *and* $\inf(cA) = c(\sup A)$.
- *If $c = 0$, then* $\sup(cA) = \inf(cA) = 0$.

Proof. Let us first assume that $c > 0$. To prove that $\sup(cA) = c(\sup A)$, we claim that

$$\sup(cA) \leq c(\sup A) \quad \text{and} \quad \sup(cA) \geq c(\sup A).$$

For the first inequality, let $a' \in cA$ be arbitrary. Then $a' = ca$ for some $a \in A$. But since $a \leq \sup A$, we have $a' = ca \leq c(\sup A)$. This shows that $c(\sup A)$ is an upper bound of cA, hence we have $\sup(cA) \leq c(\sup A)$. The reverse inequality also follows in a similar manner. (Or notice that $c\sup A = c\sup(c^{-1}cA) \leq cc^{-1}\sup(cA) = \sup(cA)$.) Then $\inf(cA) = c(\inf A)$ also follows in the same way.

When $c < 0$, the proof goes in almost the same way, but what changes now is that multiplying c to an inequality reverses the order. I leave the detail of the proof to you. $\square$

10.2. **Refinement of partition.** Suppose that a closed bounded interval $[a, b]$ is given. If $P, P' \subseteq [a, b]$ are partitions such that $P \subseteq P'$, then we call P' a *refinement* of P. Thus any refinement of P is obtained by adding finitely many points of $[a, b]$. The next lemma shows why this concept is useful in the context of Riemann sum.

Lemma 3. *Let P, P' be partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. Then*

- *If P' is a refinement of P, then $U(f, P') \leq U(f, P)$,*
- *If P' is a refinement of P, then $L(f, P') \geq L(f, P)$.*

In other words, refining a partition makes the upper sum to become smaller and the lower sum to become bigger.

Proof. We only prove the first part, since the second part follows mutatis mutandis. Also let us first consider a very simple case where $P = \{a, b\}$ consists of only two endpoints and $P' = \{a = t_0 < \cdots < t_m = b\}$. Then it is easy to observe that, for $1 \leq i \leq m$,

$$M_i := \sup_{x \in [t_{i-1}, t_i]} f(x) \leq \sup_{x \in [a,b]} f(x) =: M.$$

Indeed, this follows since $f(x) \leq M$ for any $a \leq x \leq b$. Then it follows that

$$U(f, P') = \sum_{i=1}^{m} M_i(t_i - t_{i-1}) \leq \sum_{i=1}^{m} M(t_i - t_{i-1}) = M(b - a) = U(f, P).$$

This observation readily generalizes to arbitrary partition P and its refinement P', but a direct proof may require huge burden of notations. Instead we give a concise demonstration. Let us write $P = \{a = x_0 < \cdots < x_n = b\}$. Also we write $I_i = [x_{i-1}, x_i]$ for simplicity. Then it is easy to observe that $P' \cap I_i$ is a partition in I_i which is a refinement of $\{x_{i-1}, x_i\}$. Consequently,

$$U(f, P') = \sum_{i=1}^{n} U(f|_{I_i}, P' \cap I_i) \leq \sum_{i=1}^{n} U(f|_{I_i}, \{x_{i-1}, x_i\}) = U(f, P)$$

and the proof is done. $\square$

10.3. **Riemann integrability condition.**

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the followings are equivalent:*

(i) *f is Riemann integrable.*
(ii) *For any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Proof. (i) $\implies$ (ii) : Let $\varepsilon > 0$ be arbitrary. Using property of infimum and supremum, pick partitions P_1 and P_2 such that

$$U(f, P_1) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \quad \text{and} \quad L(f, P_2) > \underline{\int_a^b} f - \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ be the common refinement. Then by Lemma 1.3, we also have

$$U(f,P) \le U(f,P_1) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \quad \text{and} \quad L(f,P) \ge L(f,P_2) > \underline{\int_a^b} f - \frac{\varepsilon}{2}.$$

But since f is Riemann integrable, both the upper Riemann integral and the lower Riemann integral coincide, let $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$, and hence

$$U(f,P) - L(f,P) < \left(\int_a^b f + \frac{\varepsilon}{2}\right) - \left(\int_a^b f - \frac{\varepsilon}{2}\right) = \varepsilon.$$

(ii) $\implies$ (i) : For each $\varepsilon > 0$, pick a partition P satisfying the condition of (ii). Then we have

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P) - L(f,P) < \varepsilon.$$

Now since ε is arbitrary, taking $\varepsilon \to 0^+$ shows that $\overline{\int_a^b} f - \underline{\int_a^b} f = 0$, which implies (i) as desired. $\square$

Exercise 10.1. Let $a < b$ be real numbers, and let $f, g \colon [a,b] \to \mathbb{R}$ be Riemann integrable functions on $[a,b]$. Then

(i) The function $f + g$ is Riemann integrable on $[a,b]$, and $\int_a^b (f+g) = (\int_a^b f) + (\int_a^b g)$.

(iv) If $f(x) \ge 0$ for all $x \in [a,b]$, then $\int_a^b f \ge 0$.

(v) If $f(x) \ge g(x)$ for all $x \in [a,b]$, then $\int_a^b f \ge \int_a^b g$.

(vi) If there exists a real number c such that $f(x) = c$ for $x \in [a,b]$, then $\int_a^b f = c(b-a)$.

(viii) Let c be a real number such that $a < c < b$. Then $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable on $[a,c]$ and $[c,b]$ respectively, and

$$\int_a^b f = \int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]}.$$

Remark 1. Our general strategy is as follows: suppose that $f : [a,b] \to \mathbb{R}$ is bounded functions. If we can somehow figure out that there exists $I \in \mathbb{R}$ satisfying

$$I \le \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} f \le I,$$

then it follows that

$$I \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le I.$$

Thus all these inequalities boil down to equalities, and we find that (1) f is Riemann integrable and (2) $\int_a^b f = I$. In our actual proofs, our goal is to identify suitable number I.

Solution. (i) To this end, we show that

$$\underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b} (f+g) \quad \text{and} \quad \overline{\int_a^b} (f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g. \tag{7}$$

In fact, this holds for any bounded function $f, g : [a,b] \to \mathbb{R}$ as we will see from our proof. Once this is proved, then for Riemann integrable functions $f, g : [a,b] \to \mathbb{R}$, we obtain

$$\int_a^b f + \int_a^b g \leq \underline{\int_a^b}(f+g) \quad \text{and} \quad \overline{\int_a^b}(f+g) \leq \int_a^b f + \int_a^b g.$$

Therefore the conclusion follows by the remark.

So it remains to prove (7). Let P, Q be any partitions of $[a,b]$. Then $P \cup Q$ is also a partition of $[a,b]$, and thus we can write $P \cup Q = \{a = x_0 < \cdots < x_n = b\}$. Then we have

$$\sup{}_{x\in[x_{i-1},x_i]}(f(x)+g(x)) \leq \left(\sup{}_{x\in[x_{i-1},x_i]}f(x)\right) + \left(\sup{}_{x\in[x_{i-1},x_i]}g(x)\right).$$

This is a direct consequence of the following fact: for all $x \in [x_{i-1}, x_i]$,

$$f(x)+g(x) \leq \left(\sup{}_{y\in[x_{i-1},x_i]}f(y)\right) + \left(\sup{}_{z\in[x_{i-1},x_i]}g(z)\right).$$

With this, we readily observe that

$$\begin{aligned}
\overline{\int_a^b}(f+g) &\leq U(f+g, P\cup Q) \\
&= \sum_{i=1}^n \left(\sup{}_{x\in[x_{i-1},x_i]}(f(x)+g(x))\right)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n \left(\sup{}_{x\in[x_{i-1},x_i]}f(x) + \sup{}_{x\in[x_{i-1},x_i]}g(x)\right)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n \left(\sup{}_{x\in[x_{i-1},x_i]}f(x)\right)(x_i - x_{i-1}) + \sum_{i=1}^n \left(\sup{}_{x\in[x_{i-1},x_i]}g(x)\right)(x_i - x_{i-1}) \\
&= U(f, P\cup Q) + U(g, P\cup Q) \\
&\leq U(f,P) + U(g,Q),
\end{aligned}$$

where at the last inequality we exploited Lemma 1.3.[3] By taking infimum for all P and for all Q separately we obtain

$$\overline{\int_a^b}(f+g) \leq \inf{}_P U(f,P) + \inf{}_Q U(g,Q) = \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Then the second inequality of (7) follows from this. The first inequality follows exactly in the same way. (All you have to do is to replace suprema by infima, upper sums by lower sums, upper integrals by lower integrals and reverse the order of every inequality.) Therefore the proof if (i) is finished.

(iv) If $f \geq 0$, then for any partition $P = \{x_0, \ldots, x_n\}$, $\inf{}_{x\in[x_{i-1},x_i]}f(x) \geq 0$. So $L(f,P) = \sum(\inf f)\Delta x_i \geq 0$. Since f is integrable, $\int_a^b f = \sup{}_P L(f,P) \geq L(f, \{a,b\}) \geq 0$.

[3]Now the reason why we introduced two partitions is clear. We want to take infimum of $U(f,P)$ and $U(g,Q)$ separately. But if P and Q are somehow related, then it is not easy to tell whether the corresponding infimum becomes the sum of infima. Thankfully, our observations show that we can indeed decouple $U(f,P)$ and $U(g,Q)$.

(v) Apply (iv) to $f-g \geq 0$ instead. Then by (i) and (iii), we have

$$\int_a^b (f-g) \geq 0 \implies \int_a^b f - \int_a^b g \geq 0 \implies \int_a^b f \geq \int_a^b g.$$

(vi) If f is a constant function with the common value c, then for any partition $P = \{x_0, \dots, x_n\}$, $\sup_{[x_{i-1},x_i]} f = c$ and $\inf_{[x_{i-1},x_i]} f = c$. Thus $U(f,P) = \sum c(x_i - x_{i-1}) = c(b-a)$ and $L(f,P) = \sum c(x_i - x_{i-1}) = c(b-a)$. Upon taking infimum and supremum respectively, it follows that

$$\overline{\int_a^b} f = c(b-a) \quad \text{and} \quad \underline{\int_a^b} f = c(b-a).$$

Since they are equal, f is integrable and $\int_a^b f = c(b-a)$, and hence the claim follows.

(vii) Let $F(x) = f(x)$ for $x \in [a,b]$ and $F(x) = 0$ for $x \in [c,d] \setminus [a,b]$. Assume $c \leq a < b \leq d$. Let P' be a partition of $[c,d]$. Let $P = P' \cup \{a,b\}$. P is a refinement of P'. Let $P = \{c = x_0 < \cdots < x_p = a < \cdots < x_q = b < \cdots < x_n = d\}$. Then $U(F,P) = \sum_{i=1}^p (\sup_{[x_{i-1},x_i]} F)\Delta x_i + \sum_{i=p+1}^q (\sup_{[x_{i-1},x_i]} F)\Delta x_i + \sum_{i=q+1}^n (\sup_{[x_{i-1},x_i]} F)\Delta x_i$. Since $F = 0$ on $[c,a]$ and $[b,d]$, the first and third sums are 0. The middle sum is $U(f,Q)$ where $Q = P \cap [a,b]$ is a partition of $[a,b]$. Thus $U(F,P) = U(f,Q)$. Since P is a refinement of P', $U(F,P) \leq U(F,P')$. Also $\overline{\int_c^d} F = \inf_{P'} U(F,P') \leq U(F,P')$. For the specific partition P constructed from P', $\overline{\int_c^d} F \leq U(F,P) = U(f,Q)$. This holds for the partition Q of $[a,b]$ derived from P'. Can we take infimum over Q? Let Q_0 be any partition of $[a,b]$. Let $P' = Q_0 \cup \{c,d\}$. Then $P = P' \cup \{a,b\} = Q_0 \cup \{c,d\}$ is a partition of $[c,d]$. $P \cap [a,b] = Q_0$. So $U(F,P) = U(f,Q_0)$. Then $\overline{\int_c^d} F = \inf_{P''} U(F,P'') \leq U(F,P) = U(f,Q_0)$. Since this holds for any Q_0, $\overline{\int_c^d} F \leq \inf_{Q_0} U(f,Q_0) = \overline{\int_a^b} f$. Similarly, $L(F,P) = L(f,Q)$. $\underline{\int_c^d} F = \sup_{P'} L(F,P') \geq L(F,P) = L(f,Q)$. So $\underline{\int_c^d} F \geq \sup_Q L(f,Q) = \underline{\int_a^b} f$. If f is integrable on $[a,b]$, $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$. Then $\underline{\int_c^d} F \geq \int_a^b f \geq \overline{\int_c^d} F$. Since $\underline{\int_c^d} F \leq \overline{\int_c^d} F$ always holds, we must have equality. Thus F is integrable on $[c,d]$ and $\int_c^d F = \int_a^b f$.

(viii) For this problem, we utilize the equivalent formulation as in Lemma 1.4. Let $\varepsilon > 0$ be arbitrary. Then there exists a partition P on $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. We may assume that P contains c, otherwise we can replace P by the refinement $P \cup \{c\}$ which preserves the inequality $U(f, P \cup \{c\}) - L(f, P \cup \{c\}) \leq U(f,P) - L(f,P) < \varepsilon$. Now write $P = \{a = x_0 < \cdots < x_m = c < x_{m+1} < \cdots < x_n = b\}$. Using this we can define $P_1 = \{a = x_0 < \cdots < x_m = c\}$ as a partition of $[a,c]$ and likewise $P_2 = \{c = x_m < \cdots < x_n = b\}$ as a partition of $[c,b]$. Then it follows that

$$\begin{aligned} U(f|_{[a,c]}, P_1) - L(f|_{[a,c]}, P_1) &= \sum_{i=1}^m \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right)(x_i - x_{i-1}) \\ &= U(f,P) - L(f,P) < \varepsilon. \end{aligned}$$

Similarly $U(f|_{[c,b]}, P_2) - L(f|_{[c,b]}, P_2) < \varepsilon$. Therefore by Lemma 1.4, both $f|_{[a,c]}$ and $f|_{[c,b]}$ are integrable. Also, using the same setting as before, we find that

$$\int_a^c f + \int_c^b f \leq U(f|_{[a,c]}, P_1) + U(f|_{[c,b]}, P_2) = U(f, P).$$

Since f is integrable on $[a, b]$, $\int_a^b f = \inf_P U(f, P)$. Thus $\int_a^c f + \int_c^b f \leq \int_a^b f$. Similar argument shows that

$$\int_a^c f + \int_c^b f \geq L(f|_{[a,c]}, P_1) + L(f|_{[c,b]}, P_2) = L(f, P).$$

Taking supremum over P, $\int_a^c f + \int_c^b f \geq \sup_P L(f, P) = \int_a^b f$. Thus by taking $\varepsilon \to 0^+$ we obtain the equality $\int_a^b f = \int_a^c f + \int_c^b f$ as desired. □

Exercise 10.2. Let $a < b$ be real numbers. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $c \in [a, b]$. Assume that, for each $\delta > 0$, we know that f is Riemann integrable on the set $\{x \in [a, b] : |x - c| \geq \delta\}$. Then f is Riemann integrable on $[a, b]$.

Proof. For the simplicity of our proof, let us assume that $a < c < b$. For the exceptional cases $c = a$ and $c = b$, only a minor modification is needed, so we only focus on the case $a < c < b$. We use Lemma 1.4 for the proof. Choose a bound $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Let $\varepsilon > 0$ be arbitrary. Now we pick δ as follows:

$$\delta = \min\left\{\frac{\varepsilon}{4M+1}, \frac{c-a}{2}, \frac{b-c}{2}\right\}.$$

Note $\delta > 0$. By the assumption, we know that f is Riemann integrable on the set $\{x \in [a, b] : |x - c| \geq \delta\}$. Note that we can write

$$\{x \in [a, b] : |x - c| \geq \delta\} = [a, c - \delta] \cup [c + \delta, b] = I_1 \cup I_2,$$

where $I_1 = [a, c-\delta]$ and $I_2 = [c+\delta, b]$ are disjoint closed intervals. Then by invoking Exercise 1 (viii), f is integrable on I_1 and I_2. By Lemma 1.4, for each $i = 1, 2$ we can find a partition P_i of I_i such that

$$U(f|_{I_i}, P_i) - L(f|_{I_i}, P_i) < \frac{\varepsilon}{3}.$$

Now let $P = P_1 \cup P_2$. This forms a partition of $[a, b]$ if we add the points $c - \delta, c + \delta$ (if they are not already endpoints) and consider the interval $[c - \delta, c + \delta]$. More precisely, let P^* be the partition of $[a, b]$ given by the union of the points in P_1 and P_2. P^* partitions $[a, b]$ into subintervals from P_1, subintervals from P_2, and the interval $[c - \delta, c + \delta]$. We have

$$\begin{aligned} U(f, P^*) - L(f, P^*) &= (U(f|_{I_1}, P_1) - L(f|_{I_1}, P_1)) + (U(f|_{I_2}, P_2) - L(f|_{I_2}, P_2)) \\ &\quad + \left(\sup_{[c-\delta,c+\delta]} f - \inf_{[c-\delta,c+\delta]} f\right)((c + \delta) - (c - \delta)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + (M - (-M))(2\delta) \\ &= \frac{2\varepsilon}{3} + 4M\delta \\ &\leq \frac{2\varepsilon}{3} + 4M\left(\frac{\varepsilon}{4M+1}\right) \\ &< \frac{2\varepsilon}{3} + \varepsilon = \frac{5\varepsilon}{3}. \end{aligned}$$

Let's re-evaluate the choice of δ. Choose $\delta = \frac{\varepsilon}{4M+1}$ is small enough? Let's try $\delta = \frac{\varepsilon}{6M}$. If $M = 0$, $f = 0$, integrable. Assume $M > 0$. Then $\delta = \min(\frac{\varepsilon}{6M}, \frac{c-a}{2}, \frac{b-c}{2})$.

$$U(f, P^*) - L(f, P^*) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 4M\delta \leq \frac{2\varepsilon}{3} + 4M\frac{\varepsilon}{6M} = \frac{2\varepsilon}{3} + \frac{2\varepsilon}{3} = \frac{4\varepsilon}{3}.$$

Still not quite $< \varepsilon$. Let's use the text's value $\delta = \min(\frac{\varepsilon}{2014M}, \dots)$. Use $\varepsilon/3$ instead of $\varepsilon/2014$. Let partitions P_1, P_2 satisfy $U - L < \varepsilon/3$ on I_1, I_2. Choose $\delta = \min(\frac{\varepsilon}{6M}, \frac{c-a}{2}, \frac{b-c}{2})$. Then

$$U(f, P^*) - L(f, P^*) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 4M\delta \leq \frac{2\varepsilon}{3} + 4M\frac{\varepsilon}{6M} = \frac{2\varepsilon}{3} + \frac{2\varepsilon}{3} = \frac{4\varepsilon}{3}.$$

This argument shows it's $O(\varepsilon)$. To get exactly $< \varepsilon$, choose $U - L < \varepsilon/3$ on I_1, I_2, and $\delta = \min(\frac{\varepsilon}{7M}, \frac{c-a}{2}, \frac{b-c}{2})$. Then $U - L < \varepsilon/3 + \varepsilon/3 + 4M(\varepsilon/(7M)) = 2\varepsilon/3 + 4\varepsilon/7 = (14+12)\varepsilon/21 = 26\varepsilon/21$. Still $> \varepsilon$. Maybe $\delta = \varepsilon/(12M)$? $U - L < 2\varepsilon/3 + 4M(\varepsilon/(12M)) = 2\varepsilon/3 + \varepsilon/3 = \varepsilon$. Yes, choosing $\delta = \min(\frac{\varepsilon}{12M}, \frac{c-a}{2}, \frac{b-c}{2})$ (assuming $M > 0$) works. Let P_1, P_2 be partitions for I_1, I_2 such that $U(f|_{I_i}, P_i) - L(f|_{I_i}, P_i) < \varepsilon/3$. Let $P^* = P_1 \cup P_2$. Then $U(f, P^*) - L(f, P^*) < \varepsilon/3 + \varepsilon/3 + 2M(2\delta) = 2\varepsilon/3 + 4M\delta \leq 2\varepsilon/3 + 4M(\varepsilon/(12M)) = 2\varepsilon/3 + \varepsilon/3 = \varepsilon$. Therefore f is integrable by Lemma 1.4. □

Exercise 10.3. Find a function $f\colon [0,1] \to \mathbb{R}$ such that f is not Riemann integrable on $[0,1]$, but such that $|f|$ is Riemann integrable on $[0,1]$.

Proof. Define f by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ -1, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Since any interval of positive length contains both rational numbers and irrational numbers, for any partition $P = \{a = x_0, \dots, x_n = b\}$ we have

$$\sup{}_{x \in [x_{i-1}, x_i]} f(x) = 1 \quad \text{and} \quad \inf{}_{x \in [x_{i-1}, x_i]} f(x) = -1.$$

Consequently it follows that $U(f, P) = \sum 1 \cdot (x_i - x_{i-1}) = b - a = 1$ and $L(f, P) = \sum (-1) \cdot (x_i - x_{i-1}) = -(b-a) = -1$ for any partition P of $[0,1]$. Hence we have

$$\overline{\int_0^1} f = 1 \neq -1 = \underline{\int_0^1} f.$$

So f is not Riemann integrable on $[0,1]$. On the other hand, $|f(x)| = 1$ for all $x \in [0,1]$. Since $|f|$ is a constant function, it is Riemann integrable on $[0,1]$ by Exercise 1(vi), and $\int_0^1 |f| dx = 1(1-0) = 1$. □

Exercise 10.4. Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. So, there exists a real number M such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let P be a partition of $[a,b]$.

- Using the identity $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$, where $\alpha, \beta \in \mathbb{R}$, show that
$$U(f^2, P) - L(f^2, P) \leq 2M(U(f, P) - L(f, P)).$$
- Show that if f is Riemann integrable on $[a,b]$, then f^2 is also Riemann integrable on $[a,b]$.

- Let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions on $[a,b]$. Using the identity $4\alpha\beta = (\alpha+\beta)^2 - (\alpha-\beta)^2$, where $\alpha, \beta \in \mathbb{R}$, show that fg is Riemann integrable on $[a,b]$.

Proof. (i) Let $P = \{a = x_0 < \cdots < x_n = b\}$. Let $\omega_i(h) = \sup_{x\in[x_{i-1},x_i]} h(x) - \inf_{x\in[x_{i-1},x_i]} h(x)$ denote the oscillation of a function h on the i-th subinterval. Then $U(h,P) - L(h,P) = \sum_{i=1}^n \omega_i(h)(x_i - x_{i-1})$. We need to show $\omega_i(f^2) \le 2M\omega_i(f)$ for each i. Let $x, y \in [x_{i-1}, x_i]$. Then

$$|f(y)^2 - f(x)^2| = |(f(y) - f(x))(f(y) + f(x))| = |f(y) - f(x)||f(y) + f(x)|.$$

Since $|f(x)| \le M$ for all x, $|f(y) + f(x)| \le |f(y)| + |f(x)| \le M + M = 2M$. Also, $f(y) - f(x) \le \sup f - \inf f = \omega_i(f)$ and $f(x) - f(y) \le \sup f - \inf f = \omega_i(f)$. So $|f(y) - f(x)| \le \omega_i(f)$. Combining these, we get $|f(y)^2 - f(x)^2| \le \omega_i(f)(2M) = 2M\omega_i(f)$. Since this holds for all $x, y \in [x_{i-1}, x_i]$, we have $\omega_i(f^2) = \sup_{x,y} |f(y)^2 - f(x)^2| \le 2M\omega_i(f)$. (Note: $\omega_i(h) = \sup_{x,y\in I_i} |h(y) - h(x)|$). Now, multiply by $(x_i - x_{i-1})$ and sum over i:

$$U(f^2,P) - L(f^2,P) = \sum_{i=1}^n \omega_i(f^2)\Delta x_i \le \sum_{i=1}^n 2M\omega_i(f)\Delta x_i = 2M\sum_{i=1}^n \omega_i(f)\Delta x_i = 2M(U(f,P) - L(f,P)).$$

(ii) Assume f is Riemann integrable. By Lemma 1.4, for any $\varepsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2M+1}$ (if $M = 0$, $f = 0$, $f^2 = 0$, integrable; assume $M > 0$). Then by (i),

$$U(f^2,P) - L(f^2,P) \le 2M(U(f,P) - L(f,P)) < 2M\left(\frac{\varepsilon}{2M+1}\right) < \varepsilon.$$

By Lemma 1.4, f^2 is also Riemann integrable on $[a,b]$.

(iii) Let f, g be Riemann integrable. By Exercise 1(i), $f + g$ is Riemann integrable. By Exercise 1(iii), $f - g$ is Riemann integrable. By part (ii), $(f+g)^2$ and $(f-g)^2$ are Riemann integrable. Using Exercise 1(iii) again, the difference $(f+g)^2 - (f-g)^2$ is Riemann integrable. Using Exercise 1(ii), the function $\frac{1}{4}((f+g)^2 - (f-g)^2)$ is Riemann integrable. But $4fg = (f+g)^2 - (f-g)^2$. So $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$ is Riemann integrable. ☐

Exercise 10.5. Let $f\colon [0,1] \to [0,\infty)$ be a continuous function such that $\int_0^1 f = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Proof. Assume otherwise. Then there exists $c \in [0,1]$ such that $f(c) > 0$. Since f is continuous, for $\varepsilon = f(c)/2 > 0$, there exists $\delta > 0$ such that if $x \in [0,1]$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon = f(c)/2$. This implies $f(x) > f(c) - \varepsilon = f(c) - f(c)/2 = f(c)/2$ for all $x \in (c-\delta, c+\delta) \cap [0,1]$. Let $[p,q]$ be a closed interval contained in $(c-\delta, c+\delta) \cap [0,1]$ with $p < q$. For instance, let $p = \max(0, c - \delta/2)$ and $q = \min(1, c + \delta/2)$. Then $q - p > 0$. On $[p,q]$, we have $f(x) \ge f(c)/2 > 0$. Since $f(x) \ge 0$ for all $x \in [0,1]$, we have

$$\int_0^1 f dx = \int_0^p f dx + \int_p^q f dx + \int_q^1 f dx.$$

Since $f \ge 0$, $\int_0^p f dx \ge 0$ and $\int_q^1 f dx \ge 0$. For the middle integral, $\int_p^q f dx \ge \int_p^q (f(c)/2)dx = (f(c)/2)(q-p) > 0$. Thus, $\int_0^1 f dx \ge (f(c)/2)(q-p) > 0$. This contradicts the given condition $\int_0^1 f = 0$. Therefore, our assumption must be false, and $f(x) = 0$ for all $x \in [0,1]$.

Another proof. Here is another proof which is in some sense a sledgehammer method. Since f is continuous, $F(x) = \int_0^x f(t)dt$ defines a differentiable function such that $F'(x) = f(x)$ on $[0, 1]$ by the Fundamental Theorem of Calculus. Moreover, since f is non-negative ($f(x) \geq 0$), for $0 \leq x \leq y \leq 1$, $F(y) - F(x) = \int_x^y f(t)dt \geq 0$. Thus $F(x)$ is non-decreasing. We have $F(0) = \int_0^0 f = 0$. We are given $F(1) = \int_0^1 f = 0$. Since F is non-decreasing and $F(0) = F(1) = 0$, it must be that $F(x) = 0$ for all $x \in [0, 1]$. This shows that F is identically zero, hence $F'(x) = f(x) = 0$ for all $x \in [0, 1]$ as well. □

Exercise 10.6. The following exercise deals with metric properties of the space of Riemann integrable functions.

- Let α, β be real numbers. Prove that $\alpha\beta \leq (\alpha^2 + \beta^2)/2$. Now, let $a < b$ be real numbers, and let $f, g\colon [a, b] \to \mathbb{R}$ be two Riemann integrable functions. Assume that $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$. (Recall that since f, g are Riemann integrable, we know that f^2, g^2 and fg are also Riemann integrable by Exercise 10.4.) Prove that
$$\int_a^b fg \leq 1.$$
- Let $a < b$ be real numbers, and let $f, g\colon [a, b] \to \mathbb{R}$ be two Riemann integrable functions. Prove the Cauchy-Schwarz inequality:
$$\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}$$
- Let $a < b$ be real numbers, and let $f, g, h\colon [a, b] \to \mathbb{R}$ be Riemann integrable functions. Define
$$d(f, g) := \left(\int_a^b (f - g)^2\right)^{1/2}.$$
Prove the triangle inequality for d. That is, show that
$$d(f, g) \leq d(f, h) + d(h, g).$$

Proof. (i) By expanding the trivial inequality $(\alpha - \beta)^2 \geq 0$, we get $\alpha^2 - 2\alpha\beta + \beta^2 \geq 0$, which implies $\alpha^2 + \beta^2 \geq 2\alpha\beta$, or $\alpha\beta \leq (\alpha^2 + \beta^2)/2$. Applying this pointwise for $f(x)$ and $g(x)$, we have $f(x)g(x) \leq (f(x)^2 + g(x)^2)/2$ for all $x \in [a, b]$. Since f, g are integrable, f^2, g^2, fg are integrable. By linearity and monotonicity of the integral (Exercise 1(ii, v)):
$$\int_a^b fgdx \leq \int_a^b \frac{f(x)^2 + g(x)^2}{2}dx = \frac{1}{2}\left(\int_a^b f^2dx + \int_a^b g^2dx\right).$$
Given $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$, we get
$$\int_a^b fg \leq \frac{1}{2}(1 + 1) = 1.$$

(ii) Let $A = \int_a^b f^2$ and $B = \int_a^b g^2$. Case 1: $A > 0$ and $B > 0$. Define $\tilde{f}(x) = f(x)/\sqrt{A}$ and $\tilde{g}(x) = g(x)/\sqrt{B}$. Then $\int_a^b \tilde{f}^2 = \int_a^b (f^2/A) = (1/A)\int_a^b f^2 = A/A = 1$. Similarly $\int_a^b \tilde{g}^2 = 1$. By part (i), $\int_a^b \tilde{f}\tilde{g} \leq 1$. Substituting back:
$$\int_a^b \frac{f(x)}{\sqrt{A}}\frac{g(x)}{\sqrt{B}}dx \leq 1 \implies \frac{1}{\sqrt{A}\sqrt{B}}\int_a^b fgdx \leq 1 \implies \int_a^b fgdx \leq \sqrt{A}\sqrt{B}.$$

This is $\int_a^b fg \leq (\int_a^b f^2)^{1/2}(\int_a^b g^2)^{1/2}$. Now apply this result to $-f$ and g. Since $\int(-f)^2 = \int f^2 = A$, we get $\int_a^b(-f)g \leq \sqrt{A}\sqrt{B}$, which means $-\int_a^b fg \leq \sqrt{A}\sqrt{B}$. Combining the two inequalities gives $|\int_a^b fg| \leq \sqrt{A}\sqrt{B}$.

Case 2: $A = \int_a^b f^2 = 0$ or $B = \int_a^b g^2 = 0$. Suppose $A = 0$. Since $f^2(x) \geq 0$ and f^2 is integrable (by Exercise 4ii), $\int_a^b f^2 = 0$ implies $f(x)^2 = 0$ "almost everywhere". If f is continuous, this implies $f(x) = 0$ for all x (by Exercise 5 logic). Then $fg = 0$, so $\int fg = 0$. The right hand side is $A^{1/2}B^{1/2} = 0^{1/2}B^{1/2} = 0$. So $0 \leq 0$ holds. For integrable functions, does $\int f^2 = 0$ imply $\int fg = 0$? We can show this using the inequality $|fg| \leq \frac{1}{2}(\varepsilon f^2 + \frac{1}{\varepsilon}g^2)$. Then $|\int fg| \leq \int |fg| \leq \frac{1}{2}(\varepsilon \int f^2 + \frac{1}{\varepsilon}\int g^2) = \frac{1}{2}(0 + \frac{1}{\varepsilon}B)$. This holds for all $\varepsilon > 0$. Letting $\varepsilon \to \infty$, this suggests $|\int fg| \leq 0$, so $\int fg = 0$. Thus the inequality holds $0 \leq 0$.

(iii) We want to show $d(f,g) \leq d(f,h) + d(h,g)$. This is equivalent to showing $d(f,g)^2 \leq (d(f,h) + d(h,g))^2$ since distances are non-negative.

$$\begin{aligned} d(f,g)^2 &= \int_a^b (f-g)^2 dx = \int_a^b ((f-h)+(h-g))^2 dx \\ &= \int_a^b [(f-h)^2 + 2(f-h)(h-g) + (h-g)^2] dx \\ &= \int_a^b (f-h)^2 dx + 2\int_a^b (f-h)(h-g)dx + \int_a^b (h-g)^2 dx \quad \text{(by linearity)} \\ &= d(f,h)^2 + 2\int_a^b (f-h)(h-g)dx + d(h,g)^2 \end{aligned}$$

Now apply the Cauchy-Schwarz inequality (part ii) to the middle term with functions $F = f - h$ and $G = h - g$:

$$\left|\int_a^b (f-h)(h-g)dx\right| \leq \left(\int_a^b (f-h)^2 dx\right)^{1/2} \left(\int_a^b (h-g)^2 dx\right)^{1/2} = d(f,h)d(h,g).$$

So, $\int_a^b (f-h)(h-g)dx \leq d(f,h)d(h,g)$. Substituting this back into the expression for $d(f,g)^2$:

$$\begin{aligned} d(f,g)^2 &\leq d(f,h)^2 + 2d(f,h)d(h,g) + d(h,g)^2 \\ &= (d(f,h) + d(h,g))^2. \end{aligned}$$

Taking the square root of both sides (which preserves the inequality since both sides are non-negative) gives the desired triangle inequality:

$$d(f,g) \leq d(f,h) + d(h,g).$$

□

11. Homework 11

Exercise 11.1. Let $(X, \|\cdot\|)$ be a normed linear space. Define $d\colon X \times X \to \mathbb{R}$ by $d(x,y) := \|x - y\|$. Show that (X, d) is a metric space.

Solution. Proof. It's clear that $d \geq 0$. If $d(x,y) = 0$, then since $||\cdot||$ is a norm it follows that $x - y = 0$ or, equivalently, that $x = y$. Now, d is symmetric insofar as

$$d(x,y) = ||x-y|| = ||-1\cdot(y-x)|| = |-1|\cdot||y-x|| = d(y,x).$$

Lastly, the triangle inequality for d follows from the triangle inequality for the norm:

$$d(x,y) = ||x-y|| = ||x-z+z-y|| \le ||x-z|| + ||z-y|| = d(x,z) + d(z,y).$$

$\square$

Exercise 11.2. Let n be a positive integer and let $x \in \mathbb{R}^n$. Show that $\|x\|_{\ell_\infty} = \lim_{p\to\infty} \|x\|_{\ell_p}$.

Solution. Proof. Fix $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Find $j \in \{1, \ldots, n\}$ with

$$||x||_{\ell^\infty(\mathbb{R}^n)} = |x_j| = ||x||_{\ell^\infty(\mathbb{R}^n)}.$$

Then for $p \ge 0$ we see that

$$\begin{aligned} ||x||_{\ell^\infty(\mathbb{R}^n)} = |x_j| &\le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n ||x||^p_{\ell^\infty(\mathbb{R}^n)}\right)^{1/p} \\ &= (n||x||^p_{\ell^\infty(\mathbb{R}^n)})^{1/p} = n^{1/p}||x||_{\ell^\infty(\mathbb{R}^n)}. \end{aligned}$$

By the squeeze theorem, $\lim_{p\to\infty} ||x||_p$ exists and equals $||x||_{\ell^\infty(\mathbb{R}^n)}$. $\square$

Exercise 11.3. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Define $\|\cdot\| : X \to [0,\infty)$ by $\|x\| := \sqrt{\langle x, x\rangle}$. Show that $(X, \|\cdot\|)$ is a normed linear space. Consequently, from Exercise 12.20, if we define $d\colon X \times X \to [0,\infty)$ by $d(x,y) := \sqrt{\langle (x-y),(x-y)\rangle}$, then (X,d) is a metric space.

Solution. We verify the norm properties. Let $\alpha \in \mathbb{F}$ (where $\mathbb{F}$ is the scalar field, $\mathbb{R}$ or $\mathbb{C}$) and $x, y \in X$.

(1) Non-negativity: $\langle x,x\rangle \ge 0$ by definition of inner product, so $\|x\| = \sqrt{\langle x,x\rangle} \ge 0$.
(2) Definiteness: $\|x\| = 0 \iff \sqrt{\langle x,x\rangle} = 0 \iff \langle x,x\rangle = 0 \iff x = 0$.
(3) Homogeneity:

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x,x\rangle} = \sqrt{|\alpha|^2\langle x,x\rangle} = \sqrt{|\alpha|^2}\sqrt{\langle x,x\rangle} = |\alpha|\|x\|.$$

(4) Triangle Inequality: We need to show $\|x+y\| \le \|x\| + \|y\|$. This is equivalent to showing $\|x+y\|^2 \le (\|x\|+\|y\|)^2$.

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2 \end{aligned}$$

By the Cauchy-Schwarz inequality ($|\langle x,y\rangle| \le \|x\|\|y\|$), we know $\mathrm{Re}\langle x,y\rangle \le |\langle x,y\rangle| \le \|x\|\|y\|$. Therefore,

$$\begin{aligned} \|x+y\|^2 &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\|+\|y\|)^2. \end{aligned}$$

Taking the square root of both sides yields $\|x+y\| \le \|x\| + \|y\|$.

All norm properties are satisfied. $\square$

Exercise 11.4. Consider the set A of all (x, y) in the plane $\mathbb{R}^2$ such that $x > 0$. Find the set of all adherent points of A, then find whether or not A is open or closed (or both, or neither).

Solution. We'll show that $\overline{\mathbb{A}} = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Let $(x, y) \in \mathbb{R}^2$. If $x > 0$, then $(x, y) \in \mathbb{A}$ and so (x, y) is an adherent point of $\mathbb{A}$ vacuously. Suppose then that $x = 0$. If $\varepsilon > 0$, then $(\varepsilon/2, y) \in \mathbb{A}$ and

$$||(x, y) - (\varepsilon/2, y)|| = ||(-\varepsilon/2, 0)|| = \frac{\varepsilon}{2} < \varepsilon$$

and so (x, y) is an adherent point of $\mathbb{A}$. If $x < 0$, then choosing $\varepsilon = -x/2$, we see that the ball of radius ε about (x, y) does not intersect $\mathbb{A}$, and so (x, y) is not an adherent point of $\mathbb{A}$. □

Exercise 11.5. Let n be a positive integer. Let $x \in \mathbb{R}^n$. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of $\mathbb{R}^n$. We write $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, so that for each $1 \leq i \leq n$, we have $x_i^{(j)} \in \mathbb{R}$, that is, $x_i^{(j)}$ is the i^{th} coordinate of $x^{(j)}$. Prove that the following three statements are equivalent.

- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_1}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_2}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_∞}.

Solution. Proof. We'll prove a more general result. From the proof in Exercise 11.2, we see that if $1 \leq p, q \leq \infty$, then

$$n^{-1/q}||x||_{\ell^p(\mathbb{R}^n)} \leq ||x||_{\ell^q(\mathbb{R}^n)} \leq n^{1/p}||x||_{\ell^p(\mathbb{R}^n)} \leq n^{1+1/p}||x||_{\ell^\infty(\mathbb{R}^n)} \leq n^{1+1/p+1/q}||x||_{\ell^q(\mathbb{R}^n)}$$

or, in short,

$$n^{-1}||x||_{\ell^p(\mathbb{R}^n)} \leq ||x||_{\ell^q(\mathbb{R}^n)} \leq n^a||x||_{\ell^p(\mathbb{R}^n)}$$

with the convention that $1/\infty = 0$. Now, say $\{x^j\}_{j=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ with x^j converging to x_0 with respect to d_p. Then $d_{\ell^p(\mathbb{R}^n)}(x_0, x^j) \to 0$ as $j \to \infty$, and so

$$d_{\ell^q(\mathbb{R}^n)}(x^j, x_0) = ||x^j - x_0||_{\ell^q(\mathbb{R}^n)} \leq n^{1/p}||x^j - x_0||_{\ell^p(\mathbb{R}^n)} = n^{1/p}d_{\ell^p(\mathbb{R}^n)}(x^j, x_0) \to 0$$

as $j \to \infty$, and so $x^j \to x_0$ with respect to d_q as $j \to \infty$. □

Exercise 11.6. Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Prove that the following statements are equivalent.

- x_0 is an adherent point of E.
- x_0 is either an interior point of E or a boundary point of E.
- There exists a sequence $(x_n)_{n=1}^{\infty}$ of elements of E which converges to x_0 with respect to the metric d.

Solution. Proof.

(i) Suppose x_0 is an interior point of X. Then $\exists r > 0$ such that $B(x_0, r) \subseteq X$. If x_0 is an exterior point, then $B(x_0, r) \cap X = \emptyset$ and so $B(x_0, r) \subseteq X^c$ which is a contradiction. Therefore x_0 is not an exterior point. If x_0 is not a boundary point, then x_0 is an exterior point or interior point. Since x_0 is not an exterior point, then x_0 is an interior point.

(ii) Suppose x_0 is not an interior point. Then for all $r > 0$ we have $B(x_0, r) \not\subseteq X$ which implies that $B(x_0, r) \cap X^c \neq \emptyset$. But also since x_0 is not a an interior point we have $B(x_0, r) \cap X \neq \emptyset$ for any $r > 0$ and so x_0 is a boundary point.

□

Exercise 11.7. Prove the following statements.

- Let (X, d) be a metric space, and let Y be a subset of X, so that $(Y, d|_{Y\times Y})$ is a metric space. If $(Y, d|_{Y\times Y})$ is complete, then Y is closed in (X, d).
- Conversely, assume that (X, d) is a complete metric space and that Y is a closed subset of X. Then $(Y, d|_{Y\times Y})$ is complete.

Solution. (i) Say $(Y, d_{Y\times Y})$ is complete. Suppose x_n is a sequence in Y converging to $x_0 \in \mathbb{X}$ with respect to d. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to $d_{Y\times Y}$. Since $(Y, d|_{Y\times Y})$ is complete, there exists some $y_0 \in Y$ with $x_n \to y_0$ with respect to $d_{Y\times Y}$. Now,

$$0 \leq d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_0) = d(x_0, x_n) + d_{Y\times Y}(x_n, y_0) \to 0$$

and so by the squeeze theorem $d(x_0, y_0) = 0$. So $x_0 = y_0 \in Y$. So Y contains its limit points, and hence Y is closed. □

Exercise 11.8. Let X be a subset of the real line $\mathbb{R}$ and let I be a set. The set X is said to be **open** if and only if there exists a (possibly uncountable) collection of open intervals $\{(a_\alpha, b_\alpha)\}_{\alpha\in I}$ where $a_\alpha < b_\alpha$ are real numbers for all $\alpha \in I$, so that $X = \cup_{\alpha\in I}(a_\alpha, b_\alpha)$. Assume that X is open. Conclude that there exists a set J which is either finite or countable, and there exists a disjoint collection of open intervals $\{(c_\alpha, d_\alpha)\}_{\alpha\in J}$ which is either finite or countable, where $c_\alpha < d_\alpha$ are real numbers for all $\alpha \in J$, so that $X = \cup_{\alpha\in J}(c_\alpha, d_\alpha)$. (Hint: given any $x \in X$, consider the largest open interval that contains x and that is contained in X. Consider then the set of all such intervals, for all $x \in X$.)

Remark 2. The analogous statement for $\mathbb{R}^2$ is not true.

Solution. Let X be an open subset of $\mathbb{R}$. Define an equivalence relation on X by declaring that $x \sim y$ if the closed interval

$$[\min\{x, y\}, \max\{x, y\}] \subseteq X.$$

Fix $x \in X$. I claim that the equivalence class $[x] = \{y \in X : y \sim x\}$ is an open interval. Let's first prove it is open. Say $y \in [x]$. Since $y \in X$, there exists some $\varepsilon > 0$ so that $B(y, \varepsilon) \subseteq X$. But then

$$[\min\{y, y + r\}, \max\{y, y + r\}], [\min\{y, y - r\}, \max\{y, y - r\}] \subseteq X$$

for every $0 < r < \varepsilon/2$. So $y+r \sim y-r \sim y \sim x$ for every $0 < r < \varepsilon/2$. Hence $B(y, \varepsilon/2) \subseteq [x]$, and so $[x]$ is open. Now, let

$$a_x = \inf\{y \in X : [y, x] \subseteq X\} \quad \text{and} \quad b_x = \inf\{y \in X : [x, y] \subseteq X\}$$

It's not difficult to prove that $(a_x, b_x) \subseteq [x]$. If $y \in [x]$ and $x < y$. Then

$$[x, y] \subseteq X$$

Since X is open, there exists some $\varepsilon > 0$ so that

$$(x, y + \varepsilon) \subseteq X$$

But then $a_x < x < y < y + \varepsilon < b_x$. So $y \in (a_x, b_x)$. This holds similarly if $y \leq x$, which proves the reverse conclusion. Since equivalence classes are disjoint, and partition X, this proves that X is a disjoint union of open intervals. To see why there must be a countable number of intervals, define a map $\Phi : \mathbb{Q} \cap X \to \{[x] : x \in X\}$ by

$$\Phi(q) = [q].$$

I claim this map is surjective. Indeed, say that $x \in X$. Then $[x]$ is an interval. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some $q \in [x] \subseteq X$. So $q \in X \cap \mathbb{Q}$ and $\Phi(q) = [x]$. Since $\mathbb{Q}$ is countable and Φ is surjective, we know that $\{[x] : x \in X\}$ is at most countable, i.e. countable infinite or finite. □

12. Homework 12

Exercise 12.1. Let (X, d) be a compact metric space. Show that (X, d) is both complete and bounded. (Hint: prove each property separately, and use argument by contradiction.)

Solution. We first prove that X is bounded. Suppose for the sake of a contradiction that (X, d) is not bounded. I claim there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with the property that

$$d(x_i, x_j) \geq 1 \text{ for all } i \neq j.$$

Choose $x_1 \in X$. Since X is not bounded we know that

$$X \not\subseteq B(x_1, 1)$$

and so we may choose $x_2 \in X \setminus B(x_1, 1)$. Note that $x_2 \notin B(x_1, 1)$ and so we see that $d(x_2, x_1) \geq 1$. Since X is not bounded we know that X cannot be contained in the union of two balls

$$X \not\subseteq B(x_1, 1) \cup B(x_2, 1)$$

since otherwise we would have it contained in one large ball $X \subseteq B(x_1, 1 + d(x_1, x_2))$ (draw a picture). So there exists some $x_3 \in X \setminus (B(x_1, 1) \cup B(x_2, 1))$. Note that

$$d(x_3, x_j) \geq 1 \text{ for } j = 1, 2$$

since $x_3 \notin B(x_1, 1) \cup B(x_2, 1)$. Continue in this way, as

$$X \not\subseteq B(x_1, 1) \cup \cdots \cup B(x_{n-1}, 1)$$

we may find $x_n \notin B(x_1, 1) \cup \cdots \cup B(x_{n-1}, 1)$ and note that

$$d(x_n, x_j) \geq 1 \text{ for } 1 \leq j \leq n - 1$$

which completes our claim. Now, since (X, d) is compact the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent and hence Cauchy subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. But

$$d(x_{n_k}, x_{n_j}) \geq 1 \text{ for } i \neq j$$

and so $\{x_{n_k}\}_{k=1}^{\infty}$ cannot be Cauchy, a contradiction. We now prove that X is complete. Suppose $\{x_n\}_{n=1}^{\infty}$ is an arbitrary Cauchy sequence in X. Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a point $x_0 \in X$ with $x_{n_k} \to x_0$. We'll show that $\{x_n\}_{n=1}^{\infty}$ converges to x_0, which will complete the proof. Let $\varepsilon > 0$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy there exists some $N_1 \in \mathbb{N}$ so that

$$d(x_n, x_m) < \frac{\varepsilon}{2}$$

for $n, m \geq N_1$. Since $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x_0 there exists some $N_2 \in \mathbb{N}$ so that

$$d(x_{n_k}, x_0) < \frac{\varepsilon}{2}$$

for all $k \geq N_2$. So if we let $N = \max\{N_1, N_2\}$, then for all $k \geq N$ we see that

$$d(x_k, x_0) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and so $\{x_n\}_{n=1}^{\infty}$ converges to x_0, as desired. □

Exercise 12.2. Let n be a positive integer. Let $(\mathbb{R}^n, d)$ denote Euclidean space with the metric $d = d_{\ell_2}$ or $d = d_{\ell_1}$. Let E be a subset of $\mathbb{R}^n$. Show that E is compact if and only if E is both closed and bounded. (Hint: use Bolzano-Weierstrass in $\mathbb{R}^n$.)

Solution. If E is compact, then from Exercise 1 we know that $(E, d_E \times E)$ is bounded and complete. By Proposition 4.8, we see that E is closed in $\mathbb{R}^n$. So E is closed and bounded, as desired. Suppose that E is closed and bounded. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in E. Since E is bounded, the sequence $\{x_j\}_{j=1}^{\infty}$ is bounded. By Bolzano-Weierstrass, there exists a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ and $x_0 \in \mathbb{R}^n$ with $x_{j_k} \to x_0$. But E is closed, so $x_0 \in E$. Hence every sequence in E has a subsequence which converges to an element of E, whence it follows that E is compact, as desired. □

Exercise 12.3. Let (X, d) be a metric space, and let $K_1, K_2, \ldots$ be a sequence of nonempty compact subsets of X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots.$$

Show that the intersection $\cap_{j=1}^{\infty} K_j$ is nonempty. (Hint: first, work in the compact metric space $(K_1, d|_{K_1 \times K_1})$. Then, consider the sets $K_1 \smallsetminus K_j$ which are open in K_1. Assume for the sake of contradiction that $\cap_{j=1}^{\infty} K_j = \emptyset$. Then apply the Open Cover Characterization of compactness.)

Solution. Suppose for the sake of a contradiction that $\bigcap_{j=1}^{\infty} K_j = \emptyset$. Then

$$K_1 \subseteq X = X \smallsetminus \emptyset = X \smallsetminus (\bigcap_{j=1}^{\infty} K_j) = \bigcup_{j=1}^{\infty} X \smallsetminus K_j = X \smallsetminus K_1 \cup (\bigcup_{j=2}^{\infty} X \smallsetminus K_j)$$

and so

$$K_1 \subseteq \bigcup_{j=2}^{\infty} X \smallsetminus K_j.$$

Since each K_j is compact and hence closed, it follows that $X \smallsetminus K_j$ is open. So $\{X \smallsetminus K_j\}_{j=2}^{\infty}$ is an open cover of K_1. Since K_1 is compact, there exists a finite sub-cover $\{X \smallsetminus K_{j_m}\}_{m=1}^{n}$ so that

$$K_1 \subseteq \bigcup_{m=1}^{n} X \smallsetminus K_{j_m} = X \smallsetminus (\bigcap_{m=1}^{n} X \smallsetminus K_{j_m}) = X \smallsetminus K_{j_m}$$

So K_{j_m} is contained in its complement, and so $K_{j_m} = \emptyset$. But every K_j is non-empty, and so we have a contradiction. □

Exercise 12.4. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f \colon X \to Y$ be a function. Show that the following two statements are equivalent.

- f is continuous at x_0.
- If we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

The proof of this exercise is contained in the next one.

Exercise 12.5. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. Show that the following four statements are equivalent.

- f is continuous at x_0, for all $x_0 \in X$.
- For all $x_0 \in X$, if we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
- For all open sets W in Y, the set $f^{-1}(W) = \{x \in X\colon f(x) \in W\}$ is an open set in X.
- For all closed sets V in Y, the set $f^{-1}(V)$ is a closed set in X.

Solution.

(1) Assume (a). Say $\{x_n\}_{n=1}^{\infty}$ is a sequence in X which converges to x_0. We want to show that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$. To this end, let $\varepsilon > 0$. Since f is continuous there exists $\delta > 0$ so that

$$d(f(x_0), f(y)) < \varepsilon$$

for all $y \in X$ with $d(y, x_0) < \delta$. Since $x_n \to x_0$ there exists $N \in \mathbb{N}$ so that $d(x_n, x_0) < \delta$ for all $n > N$. But then for all $n > N$ we have

$$d(f(x_n), f(x_0)) < \varepsilon.$$

So $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$, as desired.

(2) Assume (b). Say $x_0 \in f^{-1}(W)$ so that $f(x_0) \in W$. Suppose that x_0 is not an interior point of $f^{-1}(W)$. Then for every $n \in \mathbb{N}$ we know that

$$B(x_0, 2^{-n}) \not\subseteq f^{-1}(W)$$

So there exists x_n in X with $d(x_n, x_0) < 2^{-n}$ but $f(x_n) \in X \smallsetminus W$. So $\{x_n\}_{n=1}^{\infty}$ converges to x_0. By (b), we know that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$. But $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence in $X \smallsetminus W$, which is a closed set since W is open. So this tells us that $f(x_0) \in X \smallsetminus W$. So $x_0 \notin f^{-1}(W)$, a contradiction.

(3) Assume (c). Let V be closed. Then $Y \smallsetminus V$ is open, and

$$f^{-1}(Y \smallsetminus V) = X \smallsetminus f^{-1}(V)$$

is open by (c) which implies that $f^{-1}(V)$ is closed.

(4) Assume (d). Let $x_0 \in X$ and $\varepsilon > 0$ be arbitrary. Since $B(f(x_0), \varepsilon)$ is open, its complement $Y \smallsetminus B(f(x_0), \varepsilon)$ is closed. So

$$f^{-1}(Y \smallsetminus B(f(x_0), \varepsilon)) = X \smallsetminus f^{-1}(B(f(x_0), \varepsilon))$$

is closed, which implies that $f^{-1}(B(f(x_0), \varepsilon))$ is open. Since $x_0 \in f^{-1}(B(f(x_0), \varepsilon))$, there exists $\delta > 0$ so that

$$B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$$

or, equivalently,

$$f(B(x_0,\delta)) \subseteq B(f(x_0),\varepsilon).$$

Now, if $y \in X$ and $d(x_0,y) < \delta$ then $y \in B(x_0,\delta)$ and so $f(y) \in B(f(x_0),\varepsilon)$ or, equivalently, $d(f(x_0),f(y)) < \varepsilon$. Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $d(x_0,y) < \delta$ we have $d(f(x_0),f(y)) < \varepsilon$ which, by definition, means that f is continuous.

□

Exercise 12.6. Let (X,d_X), (Y,d_Y) and (Z,d_Z) be metric spaces. Let $f\colon (X,d_X) \to (Y,d_Y)$ be a continuous function and let $g\colon (Y,d_Y) \to (Z,d_Z)$ be a continuous function. Show that $g \circ f\colon (X,d_X) \to (Z,d_Z)$ is a continuous function.

Solution. Let $W \subseteq Z$ be open. From Exercise 5, we know that $g^{-1}(W)$ is an open subset of Y since g is continuous. From Exercise 5 again, we know that $f^{-1}(g^{-1}(W))$ is open. So for every open $W \subseteq Z$ we have

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$$

is open, and so $g \circ f$ is continuous by Exercise 5. □

Exercise 12.7. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and of an open set W such that $f(W)$ is not open.

Solution. Take the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \pi$. Then $W = (0,1)$ is an open subset of $\mathbb{R}$, but $f(W) = \{\pi\}$ is not. □

Exercise 12.8. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and of a closed set W such that $f(W)$ is not closed.

Solution. Take the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2+1}$ and take $W = \mathbb{R}$. Then W is closed, but $f(W) = (0,1]$ is not closed.

□

Exercise 12.9. Let (X,d_X) and (Y,d_Y) be metric spaces. Let $f\colon (X,d_X) \to (Y,d_Y)$ be a continuous function. Suppose $K \subseteq X$ is a compact set. Show that $f(K) = \{f(x)\colon x \in K\}$ is also a compact set.

Solution. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $f(K)$. For each $n > 1$ find $x_n \in K$ with $f(x_n) = y_n$. Now $\{x_n\}_{n=1}^{\infty}$ is a sequence in the compact set K, so there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x_0 \in K$ with $x_{n_k} \to x_0$. Since f is continuous, we see that $f(x_{n_k}) = y_{n_k} \to f(x_0) \in f(K)$. So every sequence in $f(K)$ has a convergent subsequence in $f(K)$ converging to an element in $f(K)$. So $f(K)$ is compact. □

Exercise 12.10. Using the previous exercise, prove the Maximum Principle: Let K be a closed and bounded subset of $\mathbb{R}^n$, and let $f\colon K \to \mathbb{R}$ be a continuous function. Then there exist points $a, b \in K$ such that f attains its maximum at a and f attains its minimum at b. (Hint: consider the numbers $\sup_{x\in K} f(x)$ and $\inf_{x\in K} f(x)$.)

Solution. For each $n \in \mathbb{N}$

$$\sup\{f(x) : x \in K\} - 2^{-n}$$

is not an upper bound on the set $\{f(x) : x \in K\}$ and so there exists $x_n \in K$ with

$$f(x_n) \geq \sup\{f(x) : x \in K\} - 2^{-n}.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ belongs to the compact set K, and so there exists $x_0 \in K$ and a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $x_{n_k} \to x_0$. Since f is continuous we see that

$$\begin{aligned}\sup\{f(x) : x \in K\} &\geq f(x_0) = f(\lim_{k\to\infty} x_{n_k}) \\ &= \lim_{k\to\infty} f(x_{n_k}) \\ &\geq \lim_{k\to\infty} \sup\{f(x) : x \in K\} - 2^{-n_k} \\ &= \sup\{f(x) : x \in K\}\end{aligned}$$

So

$$f(x_0) = \sup\{f(x) : x \in K\}$$

and so f attains its maximum at x_0. The proof for the minimum is similar. □

THE EXERCISES BELOW WERE OPTIONAL

Exercise 12.11 (Optional)**.** Let n be a positive integer. Let $\|\cdot\|$ and let $\|\cdot\|'$ be two norms on $\mathbb{R}^n$. Prove that these norms are equivalent. That is, there exist constants $C, c > 0$ such that, for all $x \in \mathbb{R}^n$, we have $c\,\|x\|' \leq \|x\| \leq C\,\|x\|'$. Consequently, any two norms on $\mathbb{R}^n$ are equivalent. (Hint: there are a few ways to solve this problem, but it is difficult to avoid circular reasoning. Here is one way to solve the problem.

- Note that it suffices to assume that $\|x\|' = \|x\|_{\ell_\infty}$.
- Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbb{R}^n$, and prove that

$$\|x\| \leq (\sum_{i=1}^{n} \|e_i\|)\,\|x\|_{\ell_\infty}.$$

- Consider $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \|x\|$. From the previous item, f is a continuous function from $(\mathbb{R}^n, d_{\ell_\infty})$ into $\mathbb{R}$. Let S denote the unit cube $S := \{x \in \mathbb{R}^n\colon \|x\|_{\ell_\infty} = 1\}$. Using that S is compact with respect to d_{ℓ_∞}, now apply the maximum principle to f on the set S.

Remark 3. There exist infinite dimensional vector spaces with norms that are not equivalent.

Solution. Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbb{R}^n$, so that for any $x \in \mathbb{R}^n$ we may write

$$x = \sum_{i=1}^{n} x_i e_i$$

for $x_i \in \mathbb{R}$. It follows by the triangle inequality that

$$\|x\| = \left\|\sum_{i=1}^{n} x_i e_i\right\| \leq \sum_{i=1}^{n} |x_i| \cdot \|e_i\| \leq \left(\sum_{i=1}^{n} \|e_i\|\right) \|x\|_{\ell^\infty(\mathbb{R}^n)}$$

Let

$$C = \sum_{i=1}^{n} \|e_i\|.$$

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Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|$. I claim that f is continuous on $(\mathbb{R}^n, \ell^\infty(\mathbb{R}^n))$. Indeed, if $\varepsilon > 0$ then provided $\|x - y\|_{\ell^\infty(\mathbb{R}^n)} < \varepsilon/C$ we have that

$$|f(x) - f(y)| = |\|x\| - \|y\|| \leq \|x - y\| \leq C\|x - y\|_{\ell^\infty(\mathbb{R}^n)} < C\varepsilon/C = \varepsilon.$$

So f is continuous on $(\mathbb{R}^n, \ell^\infty(\mathbb{R}^n))$. Consider the restriction of f to the set $K = \{x \in \mathbb{R}^n : \|x\|_{\ell^\infty} = 1\}$. Note that K is compact by Bolzano-Weierstrass. So there exists a minimum nonzero value on K, call it c, i.e.

$$\sup{}_{x\in K} f(x) \geq c > 0.$$

Now, if $x \in \mathbb{R}^n$, then

$$\|x\| = \frac{\|x\|}{\|x\|_{\ell^\infty(\mathbb{R}^n)}}\|x\|_{\ell^\infty(\mathbb{R}^n)} = \|\frac{x}{\|x\|_{\ell^\infty(\mathbb{R}^n)}}\|\|x\|_{\ell^\infty(\mathbb{R}^n)} \geq c\|x\|_{\ell^\infty(\mathbb{R}^n)}$$

which completes the claim. □

Exercise 12.12 (Optional)**.** Determine which of the following subsets of $\mathbb{R}^2$ are compact. Justify your answers. (As usual, if we do not specify a metric on $\mathbb{R}^2$, we mean $\mathbb{R}^2$ with the standard Euclidean metric d_{ℓ_2}.)

- $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 3\}$.
- $\{(x, y) \in \mathbb{R}^2 \colon 0 \leq xy \leq 1\}$.
- $\{(1, 1/n) \in \mathbb{R}^2 \colon n \in \mathbb{N}\}$.
- $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 3\}$.
- $\{(x, y) \in \mathbb{R}^2 \colon 0 \leq x \leq 1, \, 0 \leq y \leq x^2\}$.

Solution.

(1) It is compact. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 + y^2$. Then f is continuous since for every $\varepsilon > 0$ we have that

$$\begin{aligned}
|f(x_0, y_0) - f(x, y)| &= |x^2 - x_0^2 + y^2 - y_0^2| \\
&< |x - x_0||x + x_0| + |y - y_0||y + y_0| \\
&\leq ||(x - x_0, y - y_0)||_{\ell^2(\mathbb{R}^2)}(|x + x_0| + |y + y_0|) \\
&< d_{\ell^2}((x, y), (x_0, y_0))(1 + 2|x_0| + 1 + 2|y_0|) \\
&< \varepsilon
\end{aligned}$$

provided

$$d_{\ell^2}((x, y), (x_0, y_0)) < \delta < \min\left\{1, \frac{\varepsilon}{1 + 2|x_0| + 1 + 2|y_0|}\right\}.$$

Since f is continuous and the set $\{3\}$ is a closed subset of $\mathbb{R}$ we know that

$$f^{-1}(\{3\}) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 3\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3\}$$

is a closed subset of $\mathbb{R}^2$. If $(x,y) \in f^{-1}(\{3\})$, then

$$d((0,0),(x,y)) = \sqrt{x^2+y^2} = \sqrt{3}$$

and so $f^{-1}(\{3\}) \subseteq B((0,0),\sqrt{3}+1)$. So $f^{-1}(\{3\})$ is bounded. Since it is a closed and bounded subset of $\mathbb{R}^2$, we know it is compact.

(2) It is not compact since it is not bounded. (You'd still need to prove it's unbounded).

(3) It is not compact since it is not closed. (The sequence $(1,1/n)$ converges to $(1,0) \notin \{(1,1/n) \in \mathbb{R}^2 : n \in \mathbb{N}\}$).

(4) It is not compact since it is not closed. The sequence $\{(0,3-1/n)\}_{n=1}^{\infty}$ is a sequence in $\{(x,y) : x^2+y^2 < 3\}$ that converges to $(0,3) \notin \{(x,y) : x^2+y^2 < 3\}$. So $\{(x,y) : x^2+y^2 < 3\}$ is not closed, and cannot be compact.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x$ and $g(x,y) = y^2$. Since f and g are continuous,

$$f^{-1}([0,1]) \text{ and } g^{-1}([1,\infty))$$

are closed sets, being the pre-image of a closed set under a continuous function. But then

$$\begin{aligned}\{(x,y) : 0 \le x \le 1, 0 \le y \le x^2\} &= \{(x,y) : 0 \le x \le 1\} \cap \{(x,y) : 0 \le y \le x^2\} \\ &= f^{-1}([0,1]) \cap g^{-1}([0,\infty))\end{aligned}$$

is closed, being an intersection of two closed sets. It suffices to show this set is bounded. If $(x,y) \in \{(x,y) : 0 \le x \le 1, 0 \le y \le x^2\}$, then

$$d((0,0),(x,y)) = \sqrt{x^2+y^2} < \sqrt{1+x^4} \le \sqrt{1+1} = \sqrt{2}$$

and so it is bounded as well.

□

Exercise 12.13 (Optional)**.** Let (X,d_X) and (Y,d_Y) be vector spaces. Let $f\colon X \to Y$ be a continuous function. Let E be a connected subset of X. Show that $f(E)$ is connected.

Solution. We proceed by contradiction. Suppose E is connected but $f(E)$ is disconnected. Then there exists nonempty A, B relatively open subsets of $f(E)$ with $A \cap B = \emptyset$ and $A \cup B = E$. Since f is continuous and both A and B are open, we see that $f^{-1}(A)$ and $f^{-1}(B)$ are relatively open subsets of E with

$$E = f^{-1}(f(E)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

and

$$\emptyset = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

So E is disconnected, a contradiction. □

Exercise 12.14 (Optional)**.** Using the previous exercise, prove the Intermediate Value Theorem: Let (X,d) be a metric space. Let $f\colon X \to \mathbb{R}$ be a continuous function. Let E be a connected subset of X and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$. Then there exists $c \in E$ such that $f(c) = y$.

Solution. Let $f : X \to \mathbb{R}$ be a continuous function and E a connected subset of E. Say $a, b \in E$ and without loss of generality $y \in [f(a), f(b)]$. We know from the previous exercise that $f(E)$ is connected. By Theorem 8.4 this implies since $f(a) \in f(E)$ and $f(b) \in f(E)$ that $[f(a), f(b)] \subseteq f(E)$. In particular, $y \in f(E)$. So there exists at least one $x \in E$ with $f(x) = y$, as desired. ☐

Exercise 12.15 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f\colon X \to Y$ be a function, let $x_0 \in X$ be an adherent point of E, and let $L \in Y$. Show that the following statements are equivalent.

- $\lim_{x \to x_0; x \in E} f(x) = L$.
- For any sequence $(x^{(j)})_{j=1}^{\infty}$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to L with respect to the metric d_Y.

Solution.

(1) Assume (i). Say $\{x^{(i)}\}_{i=1}^{\infty}$ is a sequence in E which converges to x_0 with respect to the metric d_X. Let $\varepsilon > 0$. Since (i) holds there exists $\delta > 0$ so that

$$d_Y(f(x), L) < \varepsilon$$

for every $x \in E$ with $d_X(x, x_0) < \delta$. Now since $\{x^{(i)}\}_{i=1}^{\infty}$ converges to x_0 there exists N sufficiently large so that $d_X(x^{(j)}, x_0) < \delta$ for all $j > N$. But then

$$d_Y(f(x^{(j)}), L) < \varepsilon$$

for all $j > N$, as desired.

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(2) Assume (ii). Say that (i) fails, so that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) \neq L.$$

Then there exists $\varepsilon > 0$ so that for every $\delta > 0$ there exists $x \in E$ with $d_X(x, x_0) < \delta$ yet $d_Y(f(x), L) > \varepsilon$. Taking $\delta = 1/n$ for each $n \in \mathbb{N}$ this produces, for every $n \in \mathbb{N}$, an $x_n \in E$ with $d_X(x_n, x_0) < 1/n$ with $d_Y(f(x_n), L) > \varepsilon$. But then $x_n \to x_0$ with respect to x_n. So by (ii), there exists $N \in \mathbb{N}$ so that for all $n > N$ we have $d_Y(f(x_n), L) < \varepsilon$, a contradiction.

☐

Exercise 12.16 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Let $x_0 \in X$. Suppose f_j converges uniformly to f on X. Suppose that, for each $j \geq 1$, we know that f_j is continuous at x_0. Show that f is also continuous at x_0. Hint: it is probably easiest to use the $\varepsilon - \delta$ definition of continuity. Once you do this, you may require the triangle inequality in the form

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_j(x)) + d_Y(f_j(x), f_j(x_0)) + d_Y(f_j(x_0), f(x_0)).$$

Solution. Let $x_0 \in X$. Let $\varepsilon > 0$. Since f_j converges uniformly to $f(x)$, there exists $N \in \mathbb{N}$ so that

$$d(f_j(x), f(x)) < \frac{\varepsilon}{3}$$

for all $j > N$ and $x \in X$. Since f_N is continuous, there exists $\delta > 0$ so that

$$d_Y(f_N(x_0), f_N(x)) < \frac{\varepsilon}{3}$$

for all $x \in X$ with $d_X(x, x_0) < \delta$. Now, by the triangle inequality it follows that for all $x \in X$ with $d_X(x, x_0) < \delta$ we have

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x_0)) + d_Y(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So f is continuous at x_0, as desired. □

Exercise 12.17 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Suppose $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X. Suppose also that, for each $j \geq 1$, we know that f_j is bounded. Show that f is also bounded.

Solution. As $(f_j)_{j=1}^{\infty}$ converges uniformly to f, there exists some N so that

$$\sup{}_{x\in X} d_Y(f_j(x), f(x)) < 1$$

for every $n > N$. Fix $x_0 \in X$. Since f_N is bounded, there exists some C so that

$$\sup{}_{x\in X} d_Y(f_N(x_0), f_N(x)) < C.$$

By the triangle inequality and the fact that f_N is bounded, we see that

$$\begin{aligned} \sup{}_{x\in X} d_Y(f_N(x_0), f(x)) &\leq \sup{}_{x\in X} d_Y(f_N(x_0), f_N(x)) + \sup{}_{x\in X} d_Y(f_N(x), f(x)) \\ &< \sup{}_{x\in X} d_Y(f_N(x_0), f_N(x)) + 1 \\ &< C + 1 \end{aligned}$$

So

$$\{f(x) : x \in X\} \subseteq B(f_N(x_0), 1)$$

which, by definition, means that $\{f(x) : x \in X\}$ is a bounded subset of Y, and hence f is a bounded function. □

Exercise 12.18 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $B(X;Y)$ denote the set of functions $f\colon X \to Y$ that are bounded. Let $f, g \in B(X;Y)$. We define the metric $d_\infty\colon B(X;Y) \times B(X;Y) \to [0, \infty)$ by

$$d_\infty(f, g) := \sup{}_{x\in X} d_Y(f(x), g(x)).$$

Show that the space $(B(X;Y), d_\infty)$ is a metric space.

Solution. It's clear that for any $f, g \in \mathcal{B}(X;Y)$ that $d_\infty(f, g) \geq 0$, $d_\infty(f, f) = 0$, and $d_\infty(f, g) = d_\infty(g, f)$. The triangle inequality for d_∞ follows from the triangle inequality for d_Y and the subadditivity of the supremum:

$$d_\infty(f, g) \leq \sup{}_{x\in X} d_Y(f(x), h(x)) + d_Y(h(x), g(x)) \leq d_\infty(f, h) + d_\infty(h, g)$$

□

Exercise 12.19 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $B(X;Y)$. Let $f \in B(X;Y)$. Show that $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X if and only if $(f_j)_{j=1}^{\infty}$ converges to f in the metric $d_{B(X;Y)}$.

Solution. Say $(f_j)_{j=1}^{\infty}$ converges uniformly to f. Let $\varepsilon > 0$. Then there exists N sufficiently large so that $d_Y(f_N(x), f(x)) < \varepsilon$ for every $x \in X$. That is,

$$d_Y(f_N(x), f(x)) = \sup{}_{x \in X} d_Y(f_N(x), f(x)) < \varepsilon.$$

So f_N converges to f in the d_∞ metric. The reverse direction is similar. $\square$

Exercise 12.20 (Optional)**.** Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. Then the space $(C(X;Y), d_{B(X;Y)}|_{C(X;Y)\times C(X;Y)})$ is a complete subspace of $B(X;Y)$. That is, every Cauchy sequence of functions in $C(X;Y)$ converges to a function in $C(X;Y)$.

Solution. It suffices to show that a Cauchy sequence of continuous functions in the d_∞ metric converges uniformly to a continuous function. Towards this end, let $(f_j)_{j=1}^{\infty}$ be a Cauchy sequence in the d_∞ metric. For a fixed $x_0 \in X$, we see that

$$d_Y(f_n(x_0), f_m(x_0)) < d_\infty(f_n, f_m)$$

and so we see that the sequence $(f_n(x_0))_{m=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, this sequence converges, and

$$\lim_{n\to\infty} f_n(x_0)$$

exists for each $x_0 \in X$. This permits us to define a function $f : X \to Y$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We need to prove that f is continuous and that f_n converges uniformly to f. Towards proving the second point, note that for any $\varepsilon > 0$ that

$$d_Y(f(x), f_n(x)) = \lim_{m\to\infty} d_Y(f_m(x), f_n(x)) < \varepsilon$$

provided n is sufficiently large. Now, if $x_0 \in X$ is fixed we see that

$$\begin{aligned} d_Y(f(x_0), f(x)) &\leq d_Y(f(x_0), f_n(x_0)) + d_Y(f_n(x_0), f_n(x)) + d_Y(f_n(x), f(x)) \\ &< 2d_\infty(f_n, f) + d_Y(f_n(x_0), f_n(x)) \end{aligned}$$

Find N sufficiently large so that $d_\infty(f_N, f) < \varepsilon/3$. Since f_N is continuous, there exists $\delta > 0$ so that $d_Y(f_N(x_0), f(x)) < \varepsilon/3$ provided $d_X(x_0, x) < \delta$. But then

$$d_Y(f(x_0), f(x)) < \varepsilon$$

provided $d_X(x_0, x) < \delta$. So f is continuous at x_0. Since x_0 was arbitrarily chosen, it follows that f is continuous. $\square$

Exercise 12.21 (Optional)**.** Let $x \in (-1, 1)$. For each integer $j \geq 1$, define $f_j(x) := x^j$. Show that the series $\sum_{j=1}^{\infty} f_j$ converges pointwise, but not uniformly, on $(-1, 1)$ to the function $f(x) = x/(1-x)$. Also, for any $0 < t < 1$, show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to f on $[-t, t]$.

Solution. The pointwise convergence follows from summing the geometric series. Since

$$\sum_{j=1}^{n} |f_j|_\infty \leq \sum_{j=1}^{\infty} |f_j|_\infty \leq n,$$

we see that $\{\sum_{j=1}^{n} f_i\}_{n=1}^{\infty}$ is a bounded sequence of functions. Suppose, for a contradiction that it converges uniformly to $\frac{1}{1-x}$. By exercise one, this implies that $\frac{1}{1-x}$ is a bounded function on $(-1,1)$. But $\frac{1}{1-x}$ is unbounded on $(-1,1)$, a contradiction. □

Exercise 12.22 (Optional)**.** Let X be a set. Show that $\|\cdot\|_{\infty}$ is a norm on the space $B(X;\mathbb{R})$.

Solution. This is very similar to a previous exercise. □

Exercise 12.23 (Optional; **Weierstrass M-test**)**.** Let (X,d) be a metric space and let $(f_j)_{j=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series (of real numbers) $\sum_{j=1}^{\infty}\|f_j\|_{\infty}$ is absolutely convergent. Show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to some continuous function $f\colon X\to\mathbb{R}$. (Hint: first, show that the partial sums $\sum_{j=1}^{J} f_j$ form a Cauchy sequence in $C(X;\mathbb{R})$. Then, use Exercise 12.20 and the completeness of the real line $\mathbb{R}$.)

Solution. It suffices to prove that the partial sums form a Cauchy sequence in $\mathcal{C}(X;\mathbb{R})$, since exercise 4 will imply the remaining conclusions. Indeed, suppose $\sum_{j=1}^{\infty}||f_j||_{\infty}$ is absolutely convergent. To this end, let $\varepsilon>0$. Since $\sum_{j=1}^{\infty}||f_j||_{\infty}$ converges, there exists $N\in\mathbb{N}$ so that

$$\sum_{j=m}^{n}||f_j||_{\infty}<\varepsilon$$

for every $n\geq m>N$. But then

$$\left\|\sum_{j=1}^{n} f_j-\sum_{j=1}^{m} f_j\right\|_{\infty}=\left\|\sum_{j=m}^{n} f_j\right\|_{\infty}\leq\sum_{j=m}^{n}||f_j||_{\infty}<\varepsilon,$$

as desired □

Exercise 12.24 (Optional)**.** Let $a<b$ be real numbers. For each integer $j\geq 1$, let $f_j\colon[a,b]\to\mathbb{R}$ be a Riemann integrable function on $[a,b]$. Suppose $\sum_{j=1}^{\infty} f_j$ converges uniformly on $[a,b]$. Then $\sum_{j=1}^{\infty} f_j$ is also Riemann integrable, and

$$\sum_{j=1}^{\infty}\int_a^b f_j=\int_a^b\sum_{j=1}^{\infty} f_j.$$

Solution. Apply Theorem 5.1 to the sequence

$$\{\sum_{j=1}^{n} f_i\}_{n=1}^{\infty}$$

Indeed, by hypothesis they converge uniformly to $\sum_{j=1}^{\infty} f_j$, and so by Theorem 5.1 this function is Riemann integrable and

$$\int_a^b\sum_{j=1}^{\infty} f_j dx=\lim_{n\to\infty}\int_a^b\sum_{j=1}^{n} f_j dx=\lim_{n\to\infty}\sum_{j=1}^{n}\int_a^b f_j dx=\sum_{j=1}^{\infty}\int_a^b f_j dx.$$

□

Exercise 12.25 (Optional)**.** Let $a < b$. For every integer $j \geq 1$, let $f_j\colon [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $(f_j)'\colon [a,b] \to \mathbb{R}$ is continuous. Assume that the derivatives $(f_j)'$ converge uniformly to a function $g\colon [a,b] \to \mathbb{R}$ as $j \to \infty$. Assume also that there exists a point $x_0 \in [a,b]$ such that $\lim_{j\to\infty} f_j(x_0)$ exists. Then the functions f_j converge uniformly to a differentiable function f as $j \to \infty$, and $f' = g$.

Solution. Let $x_0 \in [a,b]$ be such that $\lim_{j\to\infty} f_j(x_0)$ exists. By the fundamental theorem of calculus, we know that

$$f_j(x) = f_j(x_0) + \int_{x_0}^{x} f_j'(t)dt.$$

Since $\{f_j'\}_{j=1}^{\infty}$ converges uniformly to g, we know from Theorem 5.1 that

$$\lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} f_j(x_0) + \int_{x_0}^{x} g(t)dt.$$

Denote the right hand side by $h(x)$, so that the sequence $\{f_j(x)\}_{j=1}^{\infty}$ converges pointwise to $h(x)$. By the fundamental theorem of calculus, we know that $h(x)$ is differentiable and that $h'(x) = g(x)$. We need only show that f_j converges uniformly to h. Indeed,

$$\begin{aligned} ||h - f_j||_{\infty} &= |f_j(x_0) - \lim_{j\to\infty} f_j(x_0)| + |\int_{x_0}^{x} g(t) - f_j'(t)dt| \\ &< |f_j(x_0) - \lim_{j\to\infty} f_j(x_0)| + ||g - f_j'||_{\infty}(b-a). \end{aligned}$$

Since N sufficiently large so that

$$|f_j(x_0) - \lim_{j\to\infty} f_j(x_0)| < \varepsilon/2$$

and so that $||g - f_j'||_{\infty} < \frac{\varepsilon}{2(b-a)}$ for all $j > N$. But then $||h - f_j||_{\infty} < \varepsilon$ for all $j > N$, and so f_j converge uniformly to h, as desired. □

Exercise 12.26 (Optional)**.** Let $a < b$. For every integer $j \geq 1$, let $f_j\colon [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f_j'\colon [a,b] \to \mathbb{R}$ is continuous. Assume that the series of real numbers $\sum_{j=1}^{\infty} \|f_j'\|_{\infty}$ is absolutely convergent. Assume also that there exists $x_0 \in [a,b]$ such that the series of real numbers $\sum_{j=1}^{\infty} f_j(x_0)$ converges. Then the series $\sum_{j=1}^{\infty} f_j$ converges uniformly on $[a,b]$ to a differentiable function. Moreover, for all $x \in [a,b]$,

$$\frac{d}{dx}\sum_{j=1}^{\infty} f_j(x) = \sum_{j=1}^{\infty} \frac{d}{dx} f_j(x)$$

Solution. Apply the previous exercise to the sequence

$$\{\sum_{j=1}^{n} f_i\}_{n=1}^{\infty}$$

□

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