

Please provide complete and well-written solutions to the following exercises.

Due April 8, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 9

Exercise 1. Let X be a subset of \mathbf{R} , let x_0 be a limit point of X , and let $f: X \rightarrow \mathbf{R}$ be a function. If f is differentiable at x_0 , then f is also continuous at x_0 .

Exercise 2. Let X be a subset of \mathbf{R} , let x_0 be a limit point of X , let $f: X \rightarrow \mathbf{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.

- f is differentiable at x_0 on X with derivative L .
- For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then

$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon |x - x_0|.$$

Exercise 3. Let X be a subset of \mathbf{R} , let x_0 be a limit point of X , and let $f: X \rightarrow \mathbf{R}$ and $g: X \rightarrow \mathbf{R}$ be functions.

- (i) If f is constant, so that there exists $c \in \mathbf{R}$ such that $f(x) = c$, then f is differentiable at x_0 and $f'(x_0) = 0$.
- (ii) If f is the identity function, so that $f(x) = x$, then f is differentiable at x_0 and $f'(x_0) = 1$.
- (iii) If f, g are differentiable at x_0 , then $f + g$ is differentiable at x_0 , and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$. (**Sum Rule**)
- (iv) If f, g are differentiable at x_0 , then fg is differentiable at x_0 , and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$. (**Product Rule**)
- (v) If f is differentiable at x_0 , and if $c \in \mathbf{R}$, then cf is differentiable at x_0 , and $(cf)'(x_0) = cf'(x_0)$.
- (vi) If f, g are differentiable at x_0 , then $f - g$ is differentiable at x_0 , and $(f - g)'(x_0) = f'(x_0) - g'(x_0)$.
- (vii) If g is differentiable at x_0 , and if $g(x) \neq 0$ for all $x \in X$, then $1/g$ is differentiable at x_0 , and $(1/g)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}$.
- (viii) If f, g are differentiable at x_0 , and if $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable at x_0 , and

$$(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}. \quad (\text{Quotient Rule})$$

Hint: For the product rule, you may need the following identity

$$f(x)g(x) - f(x_0)g(x_0) = f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0)).$$

Exercise 4. Let X, Y be subsets of \mathbf{R} , let $x_0 \in X$ be a limit point of X , and let $y_0 \in Y$ be a limit point of Y . Let $f: X \rightarrow Y$ be a function such that $f(x_0) = y_0$ and such that f is differentiable at x_0 . Let $g: Y \rightarrow \mathbf{R}$ be a function that is differentiable at y_0 . Then the function $g \circ f: X \rightarrow \mathbf{R}$ is differentiable at x_0 , and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

(Hint: recall that it suffices to consider a sequence $(a_n)_{n=0}^{\infty}$ of elements of X converging to x_0 . Also, from Exercise 1, f is continuous, so $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.)

Exercise 5. Let $a < b$ be real numbers, and let $f: (a, b) \rightarrow \mathbf{R}$ be a function. If $x_0 \in (a, b)$, if f is differentiable at x_0 , and if f attains a local maximum or minimum at x_0 , then $f'(x_0) = 0$.

Exercise 6. Let $a < b$ be real numbers, and let $f: [a, b] \rightarrow \mathbf{R}$ be a continuous function which is differentiable on (a, b) . Assume that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$. (Hint: use Exercise 5 and the Maximum Principle.)

Exercise 7. Let X be a subset of \mathbf{R} , let x_0 be a limit point of X , and let $f: X \rightarrow \mathbf{R}$ be a function. If f is monotone increasing and if f is differentiable at x_0 , then $f'(x_0) \geq 0$. If f is monotone decreasing and if f is differentiable at x_0 , then $f'(x_0) \leq 0$.

Exercise 8. Let $a < b$ be real numbers, and let $f: [a, b] \rightarrow \mathbf{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.