Please provide complete and well-written solutions to the following exercises.

Due March 11, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 7

Exercise 1. For any $x > 0$, show that $\lim_{n\to\infty} x^{1/n} = 1$. (Hint: first, given any $\varepsilon > 0$, show that $(1+\varepsilon)^n$ has no real upper bound M, as $n \to \infty$. To prove this claim, set $x = 1/(1+\varepsilon)$ and an exercise from the previous homework. Now, with this preliminary claim, show that for any $\varepsilon > 0$ and for any real M, there exists a positive integer n such that $M^{1/n} < 1+\varepsilon$. Now, use these two claims, and consider the cases $x > 1$ and $x < 1$ separately.)

Exercise 2. Let $m \leq n < p$ be integers, let $(a_i)_{i=m}^n, (b_i)_{i=m}^n$ be a sequences of real numbers, let k be an integer, and let c be a real number. Prove:

- $$\sum_{i=m}^n (a_i + b_i) = (\sum_{i=m}^n a_i) + (\sum_{i=m}^n b_i).$$
- $$\left|\sum_{i=m}^n a_i\right| \leq \sum_{i=m}^n |a_i|\,.$$
- $$\text{If } a_i \leq b_i \text{ for all } m \leq i \leq n, \text{ then } \sum_{i=m}^n a_i \leq \sum_{i=m}^n b_i.$$

Exercise 3. Let $\sum_{n=m}^\infty a_n$ be a formal series of real numbers. Then $\sum_{n=m}^\infty a_n$ converges if and only if: for every real number $\varepsilon > 0$, there exists an integer $N \geq M$ such that, for all $p, q \geq N$,

$$\left|\sum_{n=p}^q a_n\right| < \varepsilon.$$

(Hint: recall that a sequence is convergent if and only if it is a Cauchy sequence.)

Exercise 4. Let $\sum_{n=m}^\infty a_n$ be a formal series of real numbers. If $\sum_{n=m}^\infty a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^\infty a_n$ does not converge. (Hint: use Exercise 3.)

Exercise 5. Let $\sum_{n=m}^\infty a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is convergent. Moreover,

$$\left|\sum_{n=m}^\infty a_n\right| \leq \sum_{n=m}^\infty |a_n|\,.$$

Exercise 6.

- Let $\sum_{n=m}^\infty a_n$ be a series of real numbers converging to x, and let $\sum_{n=m}^\infty b_n$ be a series of real numbers converging to y. Then $\sum_{n=m}^\infty (a_n + b_n)$ is a convergent series that converges to $x + y$. That is,
$$\sum_{n=m}^{\infty}(a_n + b_n) = (\sum_{n=m}^{\infty} a_n) + (\sum_{n=m}^{\infty} b_n).$$

Exercise 7. Let $\sum_{n=m}^\infty a_n, \sum_{n=m}^\infty b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^\infty b_n$ is convergent, then $\sum_{n=m}^\infty a_n$ is absolutely convergent. Moreover,
$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Exercise 8. For any $n \in \mathbf{N}$, define $a_n := (-1)^{n+1}/(n+1)$. Find a bijection $g\colon \mathbf{N} \to \mathbf{N}$ such that the series $\sum_{n=0}^\infty a_{g(n)}$ diverges.

Exercise 9. Let $(b_n)_{n=m}^\infty$ be a sequence of positive numbers. Then
$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n\to\infty} b_n^{1/n}.$$

Exercise 10. Let $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty$ be sequences of real numbers. Then $(a_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$. Also, if $(b_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$, and if $(c_n)_{n=0}^\infty$ is a subsequence of $(b_n)_{n=0}^\infty$, then $(c_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$.

Exercise 11. Give an example of two convergent series of real numbers $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ such that the series $\sum_{n=0}^\infty (a_n b_n)$ is not convergent.

Exercise 12. Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers, and let L be a real number.

- If the sequence $(a_n)_{n=0}^\infty$ converges to L, then every subsequence of $(a_n)_{n=0}^\infty$ converges to L.
- Conversely, if every subsequence of $(a_n)_{n=0}^\infty$ converges to L, then $(a_n)_{n=0}^\infty$ itself converges to L.