Please provide complete and well-written solutions to the following exercises.

Due March 4, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 6

Exercise 1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers that converges to a real number x. Then x is a limit point of $(a_n)_{n=m}^{\infty}$. Moreover, x is the only limit point of $(a_n)_{n=m}^{\infty}$.

Exercise 2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence. (Note that $L^+, L^- \in \mathbf{R}^*$.)

(iii) $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.
(iv) If c is any limit point of $(a_n)_{n=m}^{\infty}$, then $L^- \leq c \leq L^+$.
(v) If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. If L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.
(vi) Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.

Exercise 3. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers such that $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite. Prove:

$$\limsup_{n\to\infty}(a_n + b_n) \leq (\limsup_{n\to\infty} a_n) + (\limsup_{n\to\infty} b_n).$$

Exercise 4. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers. Assume that $a_n \leq b_n$ for all $n \geq m$. Prove:

- $\sup(a_n)_{n=m}^{\infty} \leq \sup(b_n)_{n=m}^{\infty}$.
- $\inf(a_n)_{n=m}^{\infty} \leq \inf(b_n)_{n=m}^{\infty}$.
- $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Exercise 5. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that there exists a natural number M such that, for all $n \geq M$,

$$a_n \leq b_n \leq c_n.$$

Assume that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to the same limit L. Prove that $(b_n)_{n=m}^{\infty}$ converges to L. (Hint: use the previous exercise.)

Exercise 6. Let $x, y > 0$ be positive real numbers, and let $n, m \geq 1$ be positive integers. Prove:

(i) If $y = x^{1/n}$, then $y^n = x$.

Exercise 7. Let $x, y > 0$ be positive real numbers, and let q, r be rational numbers. Prove:

(i) x^q is a positive real number.
(ii) $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.

Exercise 8. Let $-1 < x < 1$. Show that $\lim_{n\to\infty} x^n = 0$. Using the identity $(1/x^n)x^n = 1$ for $x > 1$, conclude that x^n does not converge as $n \to \infty$ for $x > 1$.