

Please provide complete and well-written solutions to the following exercises.

Due February 18, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 5

Exercise 1. Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers. Then $(a_n)_{n=0}^\infty$ is convergent if and only if $(a_n)_{n=0}^\infty$ is a Cauchy sequence. (Hint: Given a Cauchy sequence $(a_n)_{n=0}^\infty$, use that the rationals are dense in the real numbers to replace each real a_n by some rational a'_n , so that $|a_n - a'_n|$ is small. Then, ensure that the sequence $(a'_n)_{n=0}^\infty$ is a Cauchy sequence of rationals and that $(a'_n)_{n=0}^\infty$ defines a real number which is the limit of the original sequence $(a_n)_{n=0}^\infty$.)

Exercise 2. Let $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ be real convergent sequences. Let x, y be real numbers such that $x = \lim_{n \rightarrow \infty} a_n, y = \lim_{n \rightarrow \infty} b_n$.

(i) The sequence $(a_n + b_n)_{n=0}^\infty$ converges to $x + y$. That is,

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) + \left(\lim_{n \rightarrow \infty} b_n \right).$$

(ii) The sequence $(a_n b_n)_{n=0}^\infty$ converges to xy . That is,

$$\lim_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right).$$

(iii) For any real number c , the sequence $(ca_n)_{n=0}^\infty$ converges to cx . That is,

$$c \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (ca_n).$$

(iv) The sequence $(a_n - b_n)_{n=0}^\infty$ converges to $x - y$. That is,

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) - \left(\lim_{n \rightarrow \infty} b_n \right).$$

(v) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(a_n^{-1})_{n=m}^\infty$ converges to x^{-1} . That is,

$$\lim_{n \rightarrow \infty} a_n^{-1} = \left(\lim_{n \rightarrow \infty} a_n \right)^{-1}.$$

(vi) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(b_n/a_n)_{n=m}^\infty$ converges to y/x . That is,

$$\lim_{n \rightarrow \infty} (b_n/a_n) = \left(\lim_{n \rightarrow \infty} b_n \right) / \left(\lim_{n \rightarrow \infty} a_n \right).$$

(vii) Suppose $a_n \geq b_n$ for all $n \geq 0$. Then $x \geq y$.

(Hint: you can save time by using some of these statements to prove the others. For example:

(iii) follows from (ii); (iv) follows from (i); and (vi) follows from (v) and (ii).)

Exercise 3. For each natural number n , let a_n be a real number such that $|a_n| \leq 2^{-n}$. Define $b_n := a_1 + a_2 + \cdots + a_n$. Prove that the sequence $(b_n)_{n=0}^\infty$ is convergent.

Exercise 4. Let E be a subset of \mathbf{R}^* . Then the following statements hold.

- For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$.
- Let $M \in \mathbf{R}^*$ be an upper bound for E , so that $x \leq M$ for all $x \in E$. Then $\sup(E) \leq M$.
- Let $M \in \mathbf{R}^*$ be a lower bound for E , so that $x \geq M$ for all $x \in E$. Then $\inf(E) \geq M$.

(Hint: it may be helpful to break into cases concerning whether or not E contains $+\infty$ or $-\infty$.)

Exercise 5. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then $a_n \leq x$ for all $n \geq m$. Also, for any $M \in \mathbf{R}^*$ which is an upper bound for $(a_n)_{n=m}^{\infty}$ (so that $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for any $y \in \mathbf{R}^*$ such that $y < x$, there exists at least one integer n with $n \geq m$ such that $y < a_n \leq x$. (Hint: use the previous exercise.)

Exercise 6. Let $(a_n)_{n=m}^{\infty}$ be a bounded sequence of real numbers. Assume also that $(a_n)_{n=m}^{\infty}$ is monotone increasing. That is, $a_{n+1} \geq a_n$ for all $n \geq m$. Then the sequence $(a_n)_{n=m}^{\infty}$ is convergent. In fact,

$$\lim_{n \rightarrow \infty} a_n = \sup(a_n)_{n=m}^{\infty}.$$

(Hint: use the previous exercise.)