Please provide complete and well-written solutions to the following exercises.

Due February 11, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 4

Exercise 1. Show that the notion of two sets having equal cardinality is an equivalence relation. That is, for sets X, Y, Z, show

- X has the same cardinality as X.
- If X has the same cardinality as Y, then Y has the same cardinality as X.
- If X has the same cardinality as Y, and if Y has the same cardinality as Z, then X has the same cardinality as Z.

Exercise 2. Using a proof by contradiction, show that the set $\mathbf{N}$ of natural numbers is infinite.

Exercise 3. Let X be a subset of the natural numbers $\mathbf{N}$. Prove that X is at most countable.

Exercise 4. Let Y be a set. Let $f\colon \mathbf{N} \to Y$ be a function. Then $f(\mathbf{N})$ is at most countable. (Hint: consider the set $A := \{n \in \mathbf{N}\colon f(n) \neq f(m) \,\text{for all}\, 0 \leq m < n\}$. Prove that f is a bijection from A onto $f(\mathbf{N})$. Then use the previous exercise.)

Exercise 5. Let X, Y be countable sets. Show that $X \cup Y$ is a countable set.

Exercise 6. Let X, Y be countable sets. Show that $X \times Y$ is a countable set.