Please provide complete and well-written solutions to the following exercises.

Due February 4, 10AM PST, to be uploaded as a single PDF document to Brightspace.

# Homework 3

**Exercise 1.** Let $x$ be a real number and let $\varepsilon > 0$ be any rational number. Show that there exists a rational number $y$ such that $|x - y| < \varepsilon$.

**Exercise 2.** Let $x, z$ be real numbers with $x < z$. Show that there exists a rational number $y$ with $x < y < z$. (Hint: use the previous exercise, and the Archimedean property.)

**Exercise 3.** Let $x$ be a real number. Show that there exists a Cauchy sequence of rational numbers $(a_n)_{n=0}^{\infty}$ such that $x = \mathrm{LIM}_{n\to\infty} a_n$, and such that $a_n > x$ for all $n \geq 0$.

**Exercise 4.** For every real number $x$, show that exactly one of the following statements is true: $x$ is positive, $x$ is negative, or $x$ is zero. Show that if $x, y$ are positive real numbers, then $x + y$ is positive, and $xy$ is positive.

**Exercise 5.** Let $x, y$ be real numbers. Prove that $(x^2 + y^2)/2 \geq xy$.

**Exercise 6.** Let $A$ be the set of real numbers

$$A = \left\{\frac{1}{n} \colon n \geq 1,\, n \in \mathbf{N}\right\} = \{1, 1/2, 1/3, 1/4, \ldots\}.$$

Compute $\sup(A)$ and $\inf(A)$.