Please provide complete and well-written solutions to the following exercises.

Due April 29, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 12

Exercise 1. Let (X,d) be a compact metric space. Show that (X,d) is both complete and bounded. (Hint: prove each property separately, and use argument by contradiction.)

Exercise 2. Let n be a positive integer. Let $(\mathbf{R}^n, d)$ denote Euclidean space with the metric $d = d_{\ell_2}$ or $d = d_{\ell_1}$. Let E be a subset of $\mathbf{R}^n$. Show that E is compact if and only if E is both closed and bounded. (Hint: use Bolzano-Weierstrass in $\mathbf{R}^n$.)

Exercise 3. Let (X,d) be a metric space, and let $K_1, K_2, \ldots$ be a sequence of nonempty compact subsets of X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots .$$

Show that the intersection $\cap_{j=1}^{\infty} K_j$ is nonempty. (Hint: first, work in the compact metric space $(K_1, d|_{K_1 \times K_1})$. Then, consider the sets $K_1 \setminus K_j$ which are open in K_1. Assume for the sake of contradiction that $\cap_{j=1}^{\infty} K_j = \emptyset$. Then apply the Open Cover Characterization of compactness.)

Exercise 4. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f \colon X \to Y$ be a function. Show that the following two statements are equivalent.

- f is continuous at x_0.
- If we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

Exercise 5. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f \colon X \to Y$ be a function. Show that the following four statements are equivalent.

- f is continuous at x_0, for all $x_0 \in X$.
- For all $x_0 \in X$, if we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
- For all open sets W in Y, the set $f^{-1}(W) = \{x \in X \colon f(x) \in W\}$ is an open set in X.
- For all closed sets V in Y, the set $f^{-1}(V)$ is a closed set in X.

Exercise 6. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f \colon (X, d_X) \to (Y, d_Y)$ be a continuous function and let $g \colon (Y, d_Y) \to (Z, d_Z)$ be a continuous function. Show that $g \circ f \colon (X, d_X) \to (Z, d_Z)$ is a continuous function.

Exercise 7. Give an example of a continuous function $f \colon \mathbf{R} \to \mathbf{R}$ and of an open set W such that $f(W)$ is not open.

Exercise 8. Give an example of a continuous function $f: \mathbf{R} \to \mathbf{R}$ and of a closed set W such that $f(W)$ is not closed.

Exercise 9. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: (X, d_X) \to (Y, d_Y)$ be a continuous function. Suppose $K \subseteq X$ is a compact set. Show that $f(K) = \{f(x): x \in K\}$ is also a compact set.

Exercise 10. Using the previous exercise, prove the Maximum Principle: Let K be a closed and bounded subset of $\mathbf{R}^n$, and let $f: K \to \mathbf{R}$ be a continuous function. Then there exist points $a, b \in K$ such that f attains its maximum at a and f attains its minimum at b. (Hint: consider the numbers $\sup_{x \in K} f(x)$ and $\inf_{x \in K} f(x)$.)

THE QUESTIONS BELOW ARE OPTIONAL. THEY WILL NOT BE GRADED. These questions are provided as extra practice for the final exam.

Exercise 11 (Optional)**.** Let n be a positive integer. Let $||\cdot||$ and let $||\cdot||'$ be two norms on $\mathbf{R}^n$. Prove that these norms are equivalent. That is, there exist constants $C, c > 0$ such that, for all $x \in \mathbf{R}^n$, we have $c\,||x||' \leq ||x|| \leq C\,||x||'$. Consequently, any two norms on $\mathbf{R}^n$ are equivalent. (Hint: there are a few ways to solve this problem, but it is difficult to avoid circular reasoning. Here is one way to solve the problem.

- Note that it suffices to assume that $||x||' = ||x||_{\ell_\infty}$.
- Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbf{R}^n$, and prove that

$$||x|| \leq (\sum_{i=1}^{n} ||e_i||)\,||x||_{\ell_\infty}.$$

- Consider $f: \mathbf{R}^n \to \mathbf{R}$ defined by $f(x) := ||x||$. From the previous item, f is a continuous function from $(\mathbf{R}^n, d_{\ell_\infty})$ into $\mathbf{R}$. Let S denote the unit cube $S := \{x \in \mathbf{R}^n: ||x||_{\ell_\infty} = 1\}$. Using that S is compact with respect to d_{ℓ_∞}, now apply the maximum principle to f on the set S.

Remark 1. There exist infinite dimensional vector spaces with norms that are not equivalent.

Exercise 12 (Optional)**.** Determine which of the following subsets of $\mathbf{R}^2$ are compact. Justify your answers. (As usual, if we do not specify a metric on $\mathbf{R}^2$, we mean $\mathbf{R}^2$ with the standard Euclidean metric d_{ℓ_2}.)

- $\{(x, y) \in \mathbf{R}^2: x^2 + y^2 = 3\}$.
- $\{(x, y) \in \mathbf{R}^2: 0 \leq xy \leq 1\}$.
- $\{(1, 1/n) \in \mathbf{R}^2: n \in \mathbf{N}\}$.
- $\{(x, y) \in \mathbf{R}^2: x^2 + y^2 < 3\}$.
- $\{(x, y) \in \mathbf{R}^2: 0 \leq x \leq 1,\ 0 \leq y \leq x^2\}$.

Exercise 13 (Optional)**.** Let (X, d_X) and (Y, d_Y) be vector spaces. Let $f: X \to Y$ be a continuous function. Let E be a connected subset of X. Show that $f(E)$ is connected.

Exercise 14 (Optional)**.** Using the previous exercise, prove the Intermediate Value Theorem: Let (X, d) be a metric space. Let $f: X \to \mathbf{R}$ be a continuous function. Let E be a connected subset of X and let a, b be any two elements of E. Let y be a real number between $f(a)$ and

$f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$. Then there exists $c \in E$ such that $f(c) = y$.

Exercise 15 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f\colon X \to Y$ be a function, let $x_0 \in X$ be an adherent point of E, and let $L \in Y$. Show that the following statements are equivalent.

- $\lim_{x \to x_0; x \in E} f(x) = L$.
- For any sequence $(x^{(j)})_{j=1}^{\infty}$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to L with respect to the metric d_Y.

Exercise 16 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Let $x_0 \in X$. Suppose f_j converges uniformly to f on X. Suppose that, for each $j \geq 1$, we know that f_j is continuous at x_0. Show that f is also continuous at x_0. Hint: it is probably easiest to use the $\varepsilon - \delta$ definition of continuity. Once you do this, you may require the triangle inequality in the form

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_j(x)) + d_Y(f_j(x), f_j(x_0)) + d_Y(f_j(x_0), f(x_0)).$$

Exercise 17 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Suppose $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X. Suppose also that, for each $j \geq 1$, we know that f_j is bounded. Show that f is also bounded.

Exercise 18 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $B(X; Y)$ denote the set of functions $f\colon X \to Y$ that are bounded. Let $f, g \in B(X; Y)$. We define the metric $d_\infty\colon B(X; Y) \times B(X; Y) \to [0, \infty)$ by

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

Show that the space $(B(X; Y), d_\infty)$ is a metric space.

Exercise 19 (Optional)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $B(X; Y)$. Let $f \in B(X; Y)$. Show that $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X if and only if $(f_j)_{j=1}^{\infty}$ converges to f in the metric $d_{B(X;Y)}$.

Exercise 20 (Optional)**.** Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. Then the space $(C(X; Y), d_{B(X;Y)}|_{C(X;Y) \times C(X;Y)})$ is a complete subspace of $B(X; Y)$. That is, every Cauchy sequence of functions in $C(X; Y)$ converges to a function in $C(X; Y)$.

Exercise 21 (Optional)**.** Let $x \in (-1, 1)$. For each integer $j \geq 1$, define $f_j(x) := x^j$. Show that the series $\sum_{j=1}^{\infty} f_j$ converges pointwise, but not uniformly, on $(-1, 1)$ to the function $f(x) = x/(1 - x)$. Also, for any $0 < t < 1$, show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to f on $[-t, t]$.

Exercise 22 (Optional)**.** Let X be a set. Show that $||\cdot||_\infty$ is a norm on the space $B(X; \mathbf{R})$.

Exercise 23 (Optional; Weierstrass M-test)**.** Let (X, d) be a metric space and let $(f_j)_{j=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series (of real numbers) $\sum_{j=1}^{\infty} ||f_j||_\infty$ is absolutely convergent. Show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to some continuous function $f\colon X \to \mathbf{R}$. (Hint: first, show that the partial sums

$\sum_{j=1}^J f_j$ form a Cauchy sequence in $C(X; \mathbf{R})$. Then, use Exercise 20 and the completeness of the real line $\mathbf{R}$.)

Exercise 24 (Optional)**.** Let $a < b$ be real numbers. For each integer $j \geq 1$, let $f_j \colon [a, b] \to \mathbf{R}$ be a Riemann integrable function on $[a, b]$. Suppose $\sum_{j=1}^\infty f_j$ converges uniformly on $[a, b]$. Then $\sum_{j=1}^\infty f_j$ is also Riemann integrable, and

$$\sum_{j=1}^{\infty} \int_a^b f_j = \int_a^b \sum_{j=1}^{\infty} f_j.$$

Exercise 25 (Optional)**.** Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a, b] \to \mathbf{R}$ be a differentiable function whose derivative $(f_j)' \colon [a, b] \to \mathbf{R}$ is continuous. Assume that the derivatives $(f_j)'$ converge uniformly to a function $g \colon [a, b] \to \mathbf{R}$ as $j \to \infty$. Assume also that there exists a point $x_0 \in [a, b]$ such that $\lim_{j\to\infty} f_j(x_0)$ exists. Then the functions f_j converge uniformly to a differentiable function f as $j \to \infty$, and $f' = g$.

Exercise 26 (Optional)**.** Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a, b] \to \mathbf{R}$ be a differentiable function whose derivative $f_j' \colon [a, b] \to \mathbf{R}$ is continuous. Assume that the series of real numbers $\sum_{j=1}^\infty ||f_j'||_\infty$ is absolutely convergent. Assume also that there exists $x_0 \in [a, b]$ such that the series of real numbers $\sum_{j=1}^\infty f_j(x_0)$ converges. Then the series $\sum_{j=1}^\infty f_j$ converges uniformly on $[a, b]$ to a differentiable function. Moreover, for all $x \in [a, b]$,

$$\frac{d}{dx} \sum_{j=1}^{\infty} f_j(x) = \sum_{j=1}^{\infty} \frac{d}{dx} f_j(x)$$