Please provide complete and well-written solutions to the following exercises.

Due April 22, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 11

Exercise 1. Let $(X, ||\cdot||)$ be a normed linear space. Define $d\colon X \times X \to \mathbf{R}$ by $d(x,y) := ||x-y||$. Show that (X,d) is a metric space.

Exercise 2. Let n be a positive integer and let $x \in \mathbf{R}^n$. Show that $||x||_{\ell_\infty} = \lim_{p\to\infty} ||x||_{\ell_p}$.

Exercise 3. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Define $||\cdot|| : X \to [0,\infty)$ by $||x|| := \sqrt{\langle x, x\rangle}$. Show that $(X, ||\cdot||)$ is a normed linear space. Consequently, from Exercise 1, if we define $d\colon X \times X \to [0,\infty)$ by $d(x,y) := \sqrt{\langle (x-y),(x-y)\rangle}$, then (X,d) is a metric space.

Exercise 4. Consider the set A of all (x,y) in the plane $\mathbf{R}^2$ such that $x > 0$. Find the set of all adherent points of A, then find whether or not A is open or closed (or both, or neither).

Exercise 5. Let n be a positive integer. Let $x \in \mathbf{R}^n$. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of $\mathbf{R}^n$. We write $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, so that for each $1 \leq i \leq n$, we have $x_i^{(j)} \in \mathbf{R}$, that is, $x_i^{(j)}$ is the i^{th} coordinate of $x^{(j)}$. Prove that the following three statements are equivalent.

- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_1}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_2}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_∞}.

Exercise 6. Let (X,d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Prove that the following statements are equivalent.

- x_0 is an adherent point of E.
- x_0 is either an interior point of E or a boundary point of E.
- There exists a sequence $(x_n)_{n=1}^{\infty}$ of elements of E which converges to x_0 with respect to the metric d.

Exercise 7. Prove the following statements.

- Let (X,d) be a metric space, and let Y be a subset of X, so that $(Y, d|_{Y\times Y})$ is a metric space. If $(Y, d|_{Y\times Y})$ is complete, then Y is closed in (X,d).
- Conversely, assume that (X,d) is a complete metric space and that Y is a closed subset of X. Then $(Y, d|_{Y\times Y})$ is complete.

Exercise 8. Let X be a subset of the real line $\mathbf{R}$ and let I be a set. The set X is said to be **open** if and only if there exists a (possibly uncountable) collection of open intervals $\{(a_\alpha, b_\alpha)\}_{\alpha\in I}$ where $a_\alpha < b_\alpha$ are real numbers for all $\alpha \in I$, so that $X = \cup_{\alpha\in I}(a_\alpha, b_\alpha)$.

Assume that X is open. Conclude that there exists a set J which is either finite or countable, and there exists a disjoint collection of open intervals $\{(c_\alpha, d_\alpha)\}_{\alpha\in J}$ which is either finite or countable, where $c_\alpha < d_\alpha$ are real numbers for all $\alpha \in J$, so that $X = \cup_{\alpha\in J}(c_\alpha, d_\alpha)$. (Hint: given any $x \in X$, consider the largest open interval that contains x and that is contained in X. Consider then the set of all such intervals, for all $x \in X$.)

Remark 1. The analogous statement for $\mathbf{R}^2$ is not true.