

Please provide complete and well-written solutions to the following exercises.

Due April 15, 10AM PST, to be uploaded as a single PDF document to Brightspace.

Homework 10

Exercise 1. Let $a < b$ be real numbers, and let $f, g: [a, b] \rightarrow \mathbf{R}$ be Riemann integrable functions on $[a, b]$. Then

- (i) The function $f + g$ is Riemann integrable on $[a, b]$, and $\int_a^b (f + g) = (\int_a^b f) + (\int_a^b g)$.
- (iv) If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.
- (v) If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.
- (vi) If there exists a real number c such that $f(x) = c$ for $x \in [a, b]$, then $\int_a^b f = c(b - a)$.
- (viii) Let c be a real number such that $a < c < b$. Then $f|_{[a, c]}$ and $f|_{[c, b]}$ are Riemann integrable on $[a, c]$ and $[c, b]$ respectively, and

$$\int_a^b f = \int_a^c f|_{[a, c]} + \int_c^b f|_{[c, b]}.$$

Exercise 2. Let $a < b$ be real numbers. Let $f: [a, b] \rightarrow \mathbf{R}$ be a bounded function. Let $c \in [a, b]$. Assume that, for each $\delta > 0$, we know that f is Riemann integrable on the set $\{x \in [a, b] : |x - c| \geq \delta\}$. Then f is Riemann integrable on $[a, b]$.

Exercise 3. Find a function $f: [0, 1] \rightarrow \mathbf{R}$ such that f is not Riemann integrable on $[0, 1]$, but such that $|f|$ is Riemann integrable on $[0, 1]$.

Exercise 4. Let $a < b$ be real numbers. Let $f: [a, b] \rightarrow \mathbf{R}$ be a bounded function. So, there exists a real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$. Let P be a partition of $[a, b]$.

- Using the identity $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$, where $\alpha, \beta \in \mathbf{R}$, show that

$$U(f^2, P) - L(f^2, P) \leq 2M(U(f, P) - L(f, P)).$$

- Show that if f is Riemann integrable on $[a, b]$, then f^2 is also Riemann integrable on $[a, b]$.
- Let $f, g: [a, b] \rightarrow \mathbf{R}$ be Riemann integrable functions on $[a, b]$. Using the identity $4\alpha\beta = (\alpha + \beta)^2 - (\alpha - \beta)^2$, where $\alpha, \beta \in \mathbf{R}$, show that fg is Riemann integrable on $[a, b]$.

Exercise 5. Let $f: [0, 1] \rightarrow [0, \infty)$ be a continuous function such that $\int_0^1 f = 0$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Exercise 6. The following exercise deals with metric properties of the space of Riemann integrable functions.

- Let α, β be real numbers. Prove that $\alpha\beta \leq (\alpha^2 + \beta^2)/2$. Now, let $a < b$ be real numbers, and let $f, g: [a, b] \rightarrow \mathbf{R}$ be two Riemann integrable functions. Assume that $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$. (Recall that since f, g are Riemann integrable, we know that f^2, g^2 and fg are also Riemann integrable by Exercise 4.) Prove that

$$\int_a^b fg \leq 1.$$

- Let $a < b$ be real numbers, and let $f, g: [a, b] \rightarrow \mathbf{R}$ be two Riemann integrable functions. Prove the Cauchy-Schwarz inequality:

$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}$$

- Let $a < b$ be real numbers, and let $f, g, h: [a, b] \rightarrow \mathbf{R}$ be Riemann integrable functions. Define

$$d(f, g) := \left(\int_a^b (f - g)^2 \right)^{1/2}.$$

Prove the triangle inequality for d . That is, show that

$$d(f, g) \leq d(f, h) + d(h, g).$$