425 Final Solutions[1]

1. Question 1

(a) The set of rational numbers $\mathbf{Q}$ is complete.

False. $\sqrt{2}\notin\mathbf{Q}$, but there is a Cauchy sequence in $\mathbf{Q}$ that converges to $\sqrt{2}$. Consider for example a_n which is an n-digit approximation to $\sqrt{2}$.

(b) For all $x\in\mathbf{R}$, we have $-\log(1-x)=\sum_{j=1}^{\infty}x^j/j$.
(Here log denotes the natural logarithm.)

False. When $x=-2$, $-\log(1-x)=-\log(3)\in\mathbf{R}$, but the sum diverges when $x=-2$ from the ratio test, since $|a_{j+1}/a_j|=|x|\,(j+1)/j\to 2>1$ as $j\to\infty$.

(c) Let $f\colon\mathbf{R}\to\mathbf{R}$ be differentiable. Then f is continuous.

True. For any $x\in\mathbf{R}$, $f(x)-f(y)=(x-y)\frac{f(x)-f(y)}{x-y}$. Since f is differentiable $\frac{f(x)-f(y)}{x-y}\to f'(x)$ as $y\to x$, so from the product limit law $(x-y)\frac{f(x)-f(y)}{x-y}\to 0$ as $x\to y$, i.e. $f(x)-f(y)\to 0$ as $x\to y$.

(d) Let $f\colon\mathbf{R}^6\to\mathbf{R}$ be a continuous function. Let $K\subseteq\mathbf{R}^6$ be a compact set. Then $f(K)$ is compact.

True. This was a Theorem in the notes. Continuous functions map compact sets to compact sets.

(e) Let $f\colon[0,1]\to\mathbf{R}$ be a continuous function. Let $\varepsilon>0$. Then there exists a polynomial $p\colon[0,1]\to\mathbf{R}$ such that

$$\sup_{x\in[0,1]}|f(x)-p(x)|<\varepsilon.$$

True. This is the Weierstrass Approximation Theorem.

(f) Let $V=\{(x,y)\in\mathbf{R}^2\colon 1\leq x\leq 2\ \text{ or }\ 3\leq x\leq 4\}$. Then there is a continuous function $f\colon[0,1]\to V$ such that $f(0)=(1,0)$ and $f(1)=(4,0)$.

False. Suppose for the sake of contradiction that f exists as stated. $[0,1]$ is connected but V is not. If $f(0)=(1,0)$ and $f(1)=(4,0)$ then $f(V)$ is disconnected also. Since $f([0,1])$ must be connected by a Theorem from the notes (generalizing the intermediate value theorem), we get a contradiction.

2. Question 2

Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of real numbers.
Prove that $(a_n)_{n=0}^{\infty}$ is bounded.

Solution. Let $\varepsilon=1$. Then there exists $N>0$ such that for all $n,m\geq N$, we have $|a_n-a_m|<\varepsilon=1$. That is, $|a_n-a_N|\leq 1$ for all $n\geq N$. Let $A:=\max_{n=0,\ldots,N}|a_n|$. We claim that $|a_n|\leq 1+A$ for all $n\geq 0$. The case $0\leq n\leq N$ follows by definition of A. Also by definition of A, we have $|a_N|\leq A$. So, by the triangle inequality, when $n\geq N$, we have $|a_n|=|a_n-a_N|+|a_N|\leq 1+A$.

3. Question 3

Prove the following:

For any positive integer n,

$$n^3+2n$$

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is divisible by 3. (That is, show that n^3+2n is a multiple of 3.)

Solution. We induct on n. We begin with the base case $n=1$. In this case $n^3+2n = 1+2=3$ which is a multiple of 3. We now do the inductive step. Assume that n^3+2n is a multiple of 3. We need to show that $(n+1)^3+2(n+1)$ is also a multiple of 3. We have

$$(n+1)^3+2(n+1) = n^3+3n^2+3n+1+2n+2 = (n^3+2n)+3(n^2+n)+3.$$

By the inductive hypothesis n^3+2n is a multiple of 3. The remaining terms in the sum are also multiples of 3, since $3(n^2+n)+3$ is a multiple of 3. The sum of all these terms is therefore a multiple of 3. We have therefore completed the inductive step. The proof is therefore complete.

4. Question 4

Consider the set $A=\{(x,y)\in\mathbf{R}\times\mathbf{R}\colon x+y\in\mathbf{Q}\}$. Is this set finite, countable, or uncountable? Prove your assertion.

Solution. This set is uncountable. To see this, recall that the real numbers $\mathbf{R}$ are uncountable. Define a function $f\colon\mathbf{R}\to A$ by $f(x)=(x,-x)$ for all $x\in\mathbf{R}$. Note that $f(x)$ is in A for all $x\in\mathbf{R}$, since $x+(-x)=0\in\mathbf{Q}$. We now claim that f is a bijection onto its image in A. That is, if we define $f(\mathbf{R})=\{f(x)\colon x\in\mathbf{R}\}=\{(x,-x)\colon x\in\mathbf{R}\}$, then $f\colon\mathbf{R}\to f(\mathbf{R})$ is a bijection. Indeed, given any element y of $f(\mathbf{R})$ we have $y=(x,-x)$ for some $x\in\mathbf{R}$, so $f(x)=y=(x,-x)$. And this x is unique, since if $f(x)=f(x')$ for some $x,x'\in\mathbf{R}$, then $(x,-x)=(x',-x')$, so that $x=x'$. In conclusion, $f\colon\mathbf{R}\to f(\mathbf{R})$ is a bijection. We now show that A is uncountable. It cannot be the case that A is countable, since A contains the uncountable set $f(\mathbf{R})$. Similarly, A cannot be finite. Therefore, A is uncountable, as desired.

5. Question 5

Let $f\colon[0,1]\to[0,1]$ be a Riemann integrable function such that $\int_0^1 f=0$. Assume that f is continuous. Prove that $f(x)=0$ for all $x\in[0,1]$.

Solution. We argue by contradiction. Assume that $f(x)>0$ for some $x\in[0,1]$. Since f is continuous, if we choose $\varepsilon:=f(x)/2$, then there exists $\delta>0$ such that, for all $y\in[0,1]$ with $|y-x|<\delta$ such that $|f(x)-f(y)|<\varepsilon=f(x)/2$. From the reverse triangle inequality, $|f(y)|=|f(y)-f(x)+f(x)|\geq|f(x)|-|f(y)-f(x)|\geq f(x)-f(x)/2=f(x)/2>0$. That is, we have uniform lower bound on all such y. So, if P is any partition of $[0,1]$ that includes $\{x-\delta,x,x+\delta\}\cap[0,1]$, we have $L(f,P)\geq\delta f(x)/2$, by definition of $L(f,P)$. By definition of the Riemann integral, we therefore have $\int_0^1 f\geq L(f,P)\geq\delta f(x)/2>0$, a contradiction to the fact that f has integral 0. We conclude that in fact $f=0$ for all $x\in[0,1]$, as desired.

6. Question 6

Let $x\in\mathbf{R}$, and let j be a positive integer. Define the function

$$f_j(x):=\frac{x}{1+jx^2}.$$

- Show that the sequence of functions $(f_j)_{j=1}^{\infty}$ converges uniformly to some function f.
- We use the function f from the first part of the question. Show that, if $x\neq0$, then $f'(x)=\lim_{j\to\infty}f_j'(x)$. Show that, if $x=0$, then $f'(x)\neq\lim_{j\to\infty}f_j'(x)$.

Solution. Let $f(x) = 0$ for all $x \in \mathbf{R}$. Let $j > 0$, $j \in \mathbf{Z}$. Let $h_j(x) = 1/(1 + jx^2)$ for any $j > 0$, $j \in \mathbf{Z}$. Note that $\lim_{x\to\infty} h_j(x) = 0 = \lim_{x\to-\infty} h_j(x)$. Also, $h_j'(x) = -2jx/(1 + jx^2)$. That is, on the set $(-\infty, -j^{-1/4}] \cup [j^{1/4}, +\infty)$, h_j achieves its maximum value at $x = j^{-1/4}$ and at $x = -j^{-1/4}$. This maximum value is $h_j(j^{-1/4}) = 1/(1 + j^{1/2})$.

For any $x \in [-j^{-1/4}, j^{1/4}]$, we use the bound $|f_j(x)| \leq |x| \leq j^{-1/4}$, and for any other $x \in \mathbf{R}$, we use the bound $|f_j|(x) \leq 1/(1 + j^{1/2})$. That is, for any $x \in \mathbf{R}$, we have $|f_j(x)| \leq \max(j^{-1/4}, 1/(1+j^{1/2}))$. That is, for any $j > 0$, we have $d_\infty(f, f_j) \leq \max(j^{-1/4}, 1/(1+j^{1/2}))$. That is, f_j converges to f uniformly as $j \to \infty$.

(b) Show that, if $x \neq 0$, then $f'(x) = \lim_{j\to\infty} f_j'(x)$. Show that, if $x = 0$, then $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

Note: $f_j'(x) = \frac{1+jx^2-x(2jx)}{(1+jx^2)^2} = \frac{1-jx^2}{(1+jx^2)^2}$. So, if $x \neq 0$, then $\lim_{j\to\infty} f_j'(x) = \lim_{j\to\infty} \frac{-jx^2}{(1+jx^2)^2} = \lim_{j\to\infty} \frac{-jx^2}{1+2jx^2+j^2x^4} = 0$, since the numerator has a factor of j, but the denominator has a factor of j^2 (since $x \neq 0$). Since $f = 0$, we have $f'(x) = 0$, so $f'(x) = \lim_{j\to\infty} f_j'(x)$. If $x = 0$, then $f_j'(x) = 1$ for all $j > 1$, while $f'(x) = 0$, so $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

7. Question 7

Prove the first Fundamental Theorem of Calculus:

Let $a < b$ be real numbers. Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Assume that f is also differentiable on $[a, b]$, and f' is Riemann integrable on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$.

(Hint: write the Riemann sum for $\int_a^b f'$, then apply a certain Theorem to write terms of the form $f'(c_i)(x_i - x_{i-1})$ in a different form.)

Solution. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Then

$$f(b) - f(a) = f(x_n) - f(x_0) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})). \qquad (*)$$

By the Mean Value Theorem, for each $1 \leq i \leq n$ there exists $y_i \in [x_{i-1}, x_i]$ such that

$$(x_i - x_{i-1})f'(y_i) = f(x_i) - f(x_{i-1}).$$

Substituting these equalities into $(*)$, we get

$$f(b) - f(a) = \sum_{i=1}^{n} (x_i - x_{i-1})f'(y_i).$$

Applying the definitions of $L(f', P)$ and $U(f', Q)$, we have: for all partitions P, Q of $[a, b]$,

$$L(f', P) \leq f(b) - f(a) \leq U(f', Q).$$

From Definition of the lower and upper Riemann integrals, we then get

$$\underline{\int_a^b} f' \leq f(b) - f(a) \leq \overline{\int_a^b} f'. \qquad (**)$$

Since f' is Riemann integrable, $\underline{\int_a^b} f' = \overline{\int_a^b} f' = \int_a^b f'$. So, $(**)$ implies that $\int_a^b f' = f(b) - f(a)$, as desired.

8. Question 8

Let $\ell_2(\mathbf{N}) = \{(a_j)_{j=0}^{\infty} \colon \sum_{j=0}^{\infty} a_j^2 < \infty,\, a_j \in \mathbf{R}\,\forall\, j \geq 0\}$. That is, $\ell_2(\mathbf{N})$ is the set of square-summable real sequences on $\mathbf{N}$. You can freely use that $\ell_2(\mathbf{N})$ is a real inner product space, with inner product given by $\langle (a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty}\rangle := \sum_{j=0}^{\infty} a_j b_j$. From this inner product, we then obtain a norm $\left\|(a_j)_{j=0}^{\infty}\right\| := \langle (a_j)_{j=0}^{\infty}, (a_j)_{j=0}^{\infty}\rangle^{1/2} = \sqrt{\sum_{j=0}^{\infty} a_j^2}$ and associated metric on $\ell_2(\mathbf{N})$ defined by $d((a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty}) := \sqrt{\sum_{j=0}^{\infty}(a_j - b_j)^2}$. That is, $\ell_2(\mathbf{N})$ is a metric space with respect to this metric. (You can freely use this fact.)

Define

$$B(0,1) := \{(a_j)_{j=0}^{\infty} \in \ell_2(\mathbf{N}) \colon \left\|(a_j)_{j=0}^{\infty}\right\| \leq 1\}.$$

Is $B(0,1)$ compact (with respect to the metric d)? Prove your assertion.

Solution. $B(0,1)$ is not compact. It is closed and bounded but not compact. To see this, we just need to find a bounded sequence $z^{(1)}, z^{(2)}, \ldots \subseteq \ell_2(\mathbf{N})$ that has no convergent subsequence. Define $z^{(i)}$ so that $z^{(i)} = (0, \ldots, 0, 1, 0, \ldots)$, i.e. so that the i^{th} element of $z^{(i)}$ is 1 while all other elements of $z^{(i)}$ are zero. Then $\left\|z^{(i)}\right\| = 1$ for all $i \geq 1$ (so that the sequence is bounded), and $\left\|z^{(i)} - z^{(j)}\right\| = \sqrt{2}$ for all $i, j \geq 1$ with $i \neq j$. And any subsequence of $z^{(1)}, z^{(2)}, \ldots$ also has these properties. Since $\left\|z^{(i)} - z^{(j)}\right\| = \sqrt{2}$ for all $i, j \geq 1$ with $i \neq j$, this sequence (or any subsequence) cannot be a Cauchy sequence. That is, any subsequence cannot be convergent. Therefore, $B(0,1)$ is not compact.