Math 425, Spring 2025, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 15 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 120 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	88	

Reference Sheet

Let X, Y be sets. A **bijection** is a function $f \colon X \to Y$ such that, for all $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. We say that X is **finite** if and only if there exists $n \in \mathbf{N}$ such that there exists a bijection $f \colon X \to \{1 \leq i \leq n \colon i \in \mathbf{N}\}$. We say that X is **countable** if and only if there exists a bijection $f \colon X \to \mathbf{N}$. We say that X is **uncountable** if and only if X is not finite and X is not countable.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges to** L if and only if, for every real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers that is converging to a real number L. We then say that the sequence $(a_n)_{n=0}^{\infty}$ is **convergent**, and we write $L = \lim_{n\to\infty} a_n$. If $(a_n)_{n=0}^{\infty}$ is not convergent, we say that the sequence $(a_n)_{n=0}^{\infty}$ is **divergent**, and we say the limit of L is undefined.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=0}^{\infty}$ is a **Cauchy sequence** if and only if, for any real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is **bounded** if and only if there exists $M \in \mathbf{R}$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Let E be a subset of $\mathbf{R}$ with some upper bound. The least upper bound of E is called the **supremum** of E, and is denoted by $\sup(E)$ or $\sup E$. If E has no upper bound, we write $\sup(E) = +\infty$. If E is empty, we write $\sup(E) = -\infty$. Let E be a subset of $\mathbf{R}$ with some lower bound. The greatest lower bound of E is called the **infimum** of E, and is denoted by $\inf(E)$ or $\inf E$. If E has no lower bound, we write $\inf(E) = -\infty$. If E is empty, we write $\inf(E) = +\infty$.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n \colon n \geq m,\, n \in \mathbf{N}\}$. Define $\inf(a_n)_{n=m}^{\infty}$ to be the infimum of the set $\{a_n \colon n \geq m,\, n \in \mathbf{N}\}$.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers and let x be a real number. We say that x is a **limit point** of the sequence $(a_n)_{n=m}^{\infty}$ if and only if: for every real $\varepsilon > 0$, for every natural number $N \geq m$, there exists $n \geq N$ such that $|a_n - x| < \varepsilon$. We define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\geq m} \sup_{t\geq n} a_t.$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\geq m} \inf_{t\geq n} a_t.$$

Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, define the N^{th} **partial sum** S_N of this series by $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit $L \in \mathbf{R}$ as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent**, and this infinite series **converges to** L. We say that the series $\sum_{n=m}^{\infty} a_n$ is **absolutely convergent** if and only if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent. If a series is not absolutely convergent, then it is absolutely divergent.

Zero Test. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge.

Alternating Series Test. Let $(a_n)_{n=m}^{\infty}$ be a decreasing sequence of nonnegative real numbers. That is, $a_{n+1} \leq a_n$ and $a_n \geq 0$ for all $n \geq m$. Then the series $\sum_{n=m}^{\infty}(-1)^n a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$.

Comparison Test. Let $\sum_{n=m}^{\infty} a_n, \sum_{n=m}^{\infty} b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Moreover, $|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$.

The Root Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$. (i) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, the series $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. (iii) If $\alpha = 1$, no conclusion is asserted.

The Ratio Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers. (So, a_{n+1}/a_n is defined for any $n \geq m$.) (i) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. In particular, $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent.

Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is **continuous** at x_0 if and only if $\lim_{x\to x_0; x\in X} f(x) = f(x_0)$. That is, the limit of f at x_0 in X exists, and this limit is equal to $f(x_0)$. We say that f is **continuous on** X (or we just say that f is **continuous**) if and only if f is continuous at x_0 for every $x_0 \in X$. We say that f is **uniformly continuous** if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. We say that f is **Lipschitz continuous** with constant L if and only if there exists $L \geq 0$ such that, for every $x, y \in X$, we have $|f(x) - f(y)| \leq L\,|x - y|$.

Let $f\colon X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f **attains its maximum** at x_0 if and only if $f(x_0) \geq f(x)$ for all $x \in X$. We say that f **attains its minimum** at x_0 if and only if $f(x_0) \leq f(x)$ for all $x \in X$.

The Maximum Principle. Let $a < b$ be real numbers and let $f\colon [a,b] \to \mathbf{R}$ be a function that is continuous on $[a,b]$. Then f attains its maximum and minimum on $[a,b]$.

Intermediate Value Theorem. Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbf{R}$ be function that is continuous on $[a,b]$. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists a $c \in [a,b]$ such that $f(c) = y$.

Let X be a subset of $\mathbf{R}$ and let x be a real number. We say that x is a **limit point** of X if and only if, for every real $\varepsilon > 0$, there exists a $y \in X$ with $y \neq x$ such that $|y - x| < \varepsilon$.

Let X be a subset of $\mathbf{R}$, and let x_0 be an element of X which is also a limit point of X. Let $f\colon X \to \mathbf{R}$ be a function. If the limit $\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0}$ converges to a real number L, then we say that f is **differentiable** at x_0 on X **with derivative** L, and we write $f'(x_0) := L$. If this limit does not exist, or if x_0 is not a limit point of X, we leave $f'(x_0)$ undefined, and we say that f is **not differentiable** at x_0 on X.

Mean Value Theorem. Let $a < b$ be real numbers, and let $f\colon [a, b] \to \mathbf{R}$ be a continuous function which is differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Let $a < b$ be real numbers, let $f\colon [a, b] \to \mathbf{R}$ be a bounded function, and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. That is, $a = x_0 < x_1 < \cdots < x_n = b$.We define the **upper Riemann sum** $U(f, P)$ by $U(f, P) := \sum_{i=1}^{n}(\sup_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1})$. We also define the **lower Riemann sum** $L(f, P)$ by $L(f, P) := \sum_{i=1}^{n}(\inf_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1})$.

Let $a < b$ be real numbers, let $f\colon [a, b] \to \mathbf{R}$ be a bounded function. We define the **upper Riemann integral** $\overline{\int_a^b} f$ of f on $[a, b]$ by

$$\overline{\int_a^b} f := \inf\{U(f, P)\colon P \text{ is a partition of } [a, b]\}.$$

We also define the **lower Riemann integral** $\underline{\int_a^b} f$ of f on $[a, b]$ by

$$\underline{\int_a^b} f := \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\}.$$

Let $a < b$ be real numbers, let $f\colon [a, b] \to \mathbf{R}$ be a bounded function. If $\overline{\int_a^b} f = \underline{\int_a^b} f$ we say that f is **Riemann integrable** on $[a, b]$, and we define $\int_a^b f := \overline{\int_a^b} f = \underline{\int_a^b} f$.

Fundamental Theorem of Calculus, Part 1. Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Assume that f is also differentiable on $[a, b]$, and f' is Riemann integrable on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$.

Fundamental Theorem of Calculus, Part 2. Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be a Riemann integrable function. Define a function $F\colon [a, b] \to \mathbf{R}$ by $F(x) := \int_a^x f$. Then F is continuous. Moreover, if $x_0 \in [a, b]$ and if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

A **metric space** (X, d) is a set X together with a function $d\colon X \times X \to [0, \infty)$ which satisfies the following properties. (i) For all $x \in X$, we have $d(x, x) = 0$. (ii) For all $x, y \in X$ with

$x \neq y$, we have $d(x, y) > 0$. (Positivity) (iii) For all $x, y \in X$, we have $d(x, y) = d(y, x)$. (Symmetry) (iv) For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Let X be a vector space over $\mathbf{R}$. A **normed linear space** $(X, ||\cdot||)$ is a vector space X over $\mathbf{R}$ together with a norm function $||\cdot|| : X \to [0, \infty)$ which satisfies the following properties. (i) $||0|| = 0$. (ii) For all $x \in X$ with $x \neq 0$, we have $||x|| > 0$. (Positivity) (iii) For all $x \in X$ and for all $\alpha \in \mathbf{R}$, we $||\alpha x|| = |\alpha| \, ||x||$. (Homogeneity) (iv) For all $x, y \in X$, we have $||x + y|| \leq ||x|| + ||y||$. (Triangle inequality)

Let X be a vector space over $\mathbf{R}$. A **real inner product space** $(X, \langle \cdot, \cdot \rangle)$ is a vector space X over $\mathbf{R}$ together with an inner product function $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbf{R}$ which satisfies the following properties. (i) $\langle 0, 0 \rangle = 0$. (ii) For all $x \in X$ with $x \neq 0$, we have $\langle x, x \rangle > 0$. (iii) For all $x, y \in X$, we have $\langle x, y \rangle = \langle y, x \rangle$. (Symmetry) (iv) For all $x \in X$ and for all $\alpha \in \mathbf{R}$, we $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$. (Homogeneity) (v) For all $x, y, z \in X$, we have $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$. (Linearity)

Let (X, d) be a metric space. We say that (X, d) is **complete** if and only if the following property holds. For any Cauchy sequence $(x^{(j)})_{j=k}^{\infty}$ of elements of X, then there exists some $x \in X$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.

A metric space (X, d) is said to be **compact** if and only if every sequence in (X, d) has at least one convergent subsequence. We say that $Y \subseteq X$ is **compact** if and only if the metric space $(Y, d|_{Y \times Y})$ is compact.

Let (X, d) be a metric space. We say that X is **disconnected** if and only if there exist disjoint nonempty open sets V, W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a proper non-empty subset which is both open and closed.) We say that X is **connected** if and only if X is not disconnected. We say that $Y \subseteq X$ is **connected** if and only if the metric space $(Y, d|_{Y \times Y})$ is connected. We say that Y is **disconnected** if and only if the metric space $(Y, d|_{Y \times Y})$ is disconnected.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f \colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges pointwise** to f on X if and only if, for every $x \in X$, we have $\lim_{j \to \infty} f_j(x) = f(x)$. That is, for all $x \in X$, we have $\lim_{j \to \infty} d_Y(f_j(x), f(x)) = 0$. That is, for every $x \in X$ and for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$, we have $d_Y(f_j(x), f(x)) = 0$.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f \colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges uniformly** to f on X if and only if, for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$ and for all $x \in X$ we have $d_Y(f_j(x), f(x)) = 0$.

Let $a \in \mathbf{R}$ and let $r > 0$. Let E be a subset of $\mathbf{R}$ such that $(a - r, a + r) \subseteq E$. Let $f \colon E \to \mathbf{R}$. We say that the function f is **real analytic on** $(a - r, a + r)$ if and only if there exists a power series $\sum_{j=0}^{\infty} a_j (x - a)^j$ centered at a with radius of convergence R such that $R \geq r$ and such that this power series converges to f on $(a - r, a + r)$.

1. Label the following statements as TRUE or FALSE.
   If the statement is true, **EXPLAIN YOUR REASONING**.
   If the statement is false, **PROVIDE A COUNTEREXAMPLE AND/OR EXPLAIN YOUR REASONING**.

   (a) (3 points) The set of rational numbers $\mathbf{Q}$ is complete.

   TRUE FALSE (circle one)

   (b) (3 points) For all $x \in \mathbf{R}$, we have $-\log(1-x) = \sum_{j=1}^{\infty} x^j/j$.
   (Here log denotes the natural logarithm.)

   TRUE FALSE (circle one)

   (c) (3 points) Let $f\colon \mathbf{R} \to \mathbf{R}$ be differentiable. Then f is continuous.

   TRUE FALSE (circle one)

(d) (3 points) Let $f\colon \mathbf{R}^6 \to \mathbf{R}$ be a continuous function. Let $K \subseteq \mathbf{R}^6$ be a compact set. Then $f(K)$ is compact.

TRUE FALSE (circle one)

(e) (3 points) Let $f\colon [0,1] \to \mathbf{R}$ be a continuous function. Let $\varepsilon > 0$. Then there exists a polynomial $p\colon [0,1] \to \mathbf{R}$ such that

$$\sup_{x\in[0,1]} |f(x) - p(x)| < \varepsilon.$$

TRUE FALSE (circle one)

(f) (3 points) Let $V = \{(x,y) \in \mathbf{R}^2 \colon 1 \leq x \leq 2 \text{ or } 3 \leq x \leq 4\}$. Then there is a continuous function $f\colon [0,1] \to V$ such that $f(0) = (1,0)$ and $f(1) = (4,0)$.

TRUE FALSE (circle one)

2. (10 points) Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of real numbers. Prove that $(a_n)_{n=0}^{\infty}$ is bounded.

   [this was a repeated homework exercise.]

3. (10 points) Prove the following:

For any positive integer n,

$$n^3 + 2n$$

is divisible by 3. (That is, show that $n^3 + 2n$ is a multiple of 3.)

4. (10 points) Consider the set $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} \colon x + y \in \mathbf{Q}\}$. Is this set finite, countable, or uncountable? Prove your assertion.

   [this was repeated from a practice exam]

5. (10 points) Let $f\colon [0,1] \to [0,1]$ be a Riemann integrable function such that $\int_0^1 f = 0$. Assume that f is continuous. Prove that $f(x) = 0$ for all $x \in [0,1]$.

   [this was a repeated homework exercise.]

6. [this was repeated from a practice exam]

Let $x \in \mathbf{R}$, and let j be a positive integer. Define the function

$$f_j(x) := \frac{x}{1 + jx^2}.$$

(a) (5 points) Show that the sequence of functions $(f_j)_{j=1}^{\infty}$ converges uniformly to some function f.

(b) (5 points) We use the function f from the first part of the question. Show that, if $x \neq 0$, then $f'(x) = \lim_{j\to\infty} f_j'(x)$. Show that, if $x = 0$, then $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

7. (10 points) Prove the first Fundamental Theorem of Calculus:

   Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Assume that f is also differentiable on $[a, b]$, and f' is Riemann integrable on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$.

   (Hint: write the Riemann sum for $\int_a^b f'$, then apply a certain Theorem to write terms of the form $f'(c_i)(x_i - x_{i-1})$ in a different form.)

8. (10 points) Let $\ell_2(\mathbf{N}) = \{(a_j)_{j=0}^{\infty} \colon \sum_{j=0}^{\infty} a_j^2 < \infty,\ a_j \in \mathbf{R}\,\forall\, j \geq 0\}$. That is, $\ell_2(\mathbf{N})$ is the set of square-summable real sequences on $\mathbf{N}$. You can freely use that $\ell_2(\mathbf{N})$ is a real inner product space, with inner product given by $\langle (a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty} \rangle := \sum_{j=0}^{\infty} a_j b_j$. From this inner product, we then obtain a norm $\left|\left|(a_j)_{j=0}^{\infty}\right|\right| := \langle (a_j)_{j=0}^{\infty}, (a_j)_{j=0}^{\infty} \rangle^{1/2} = \sqrt{\sum_{j=0}^{\infty} a_j^2}$ and associated metric on $\ell_2(\mathbf{N})$ defined by $d((a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty}) := \sqrt{\sum_{j=0}^{\infty} (a_j - b_j)^2}$. That is, $\ell_2(\mathbf{N})$ is a metric space with respect to this metric. (You can freely use this fact.)

Define

$$B(0,1) := \{(a_j)_{j=0}^{\infty} \in \ell_2(\mathbf{N}) \colon \left|\left|(a_j)_{j=0}^{\infty}\right|\right| \leq 1\}.$$

Is $B(0,1)$ compact (with respect to the metric d)? Prove your assertion.

(Scratch paper)