425 Midterm 2 Solutions[1]

1. Question 1

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

FALSE. $\sum_{n=1}^{\infty} 1/n$ diverges, while $\lim_{n\to\infty}(1/n) = 0$, as shown in class or the notes.

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ converges.

FALSE. From the alternating series test, $\sum_{n=1}^{\infty}(-1)^n/n$ converges, but $\sum_{n=1}^{\infty} 1/n$ diverges, as shown in the notes/class.

(c) Let $f, g\colon \mathbf{R} \to \mathbf{R}$. If $\lim_{x\to 0}(f(x)+g(x))$ exists, then $\lim_{x\to 0} f(x)$ exists, and $\lim_{x\to 0} g(x))$ exists, and also
$\lim_{x\to 0}(f(x) + g(x)) = (\lim_{x\to 0} f(x)) + (\lim_{x\to 0} g(x))$.

FALSE. If $f(x) = 1/x$ and $g(x) = -1/x$ for any $x \in \mathbf{R} \setminus \{0\}$ (and $f(0) = g(0) = 0$), then the limits of f, g at 0 do not exist, but $\lim_{x\to 0}(f(x) + g(x)) = 0$.

(d) If $\sum_{n=1}^{\infty} a_n$ converges, then any rearrangement of the sum also converges. That is, for any bijection $g\colon \mathbf{N} \to \mathbf{N}$, $\sum_{n=1}^{\infty} a_{g(n)}$ converges

FALSE. We showed in class that $\sum_{n=1}^{\infty}(-1)^n/n$ converges, but it has a rearrangement that diverges.

2. Question 2

(a) Let $a_n = 1/n$ for all positive integers n.

Solution. We know that $\lim_{n\to\infty} 1/n = 0$. So, from a proposition in the notes, we have $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 0$.

(b) Let $b_n = (-1)^n$ for all positive integers n.

Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Solution. We have $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$. To see that these computations are correct, note that $a_n \leq 1$ for all $n \geq 1$, so 1 is always an upper bound for $(a_n)_{n=1}^{\infty}$. However, if $x < 1$, then for any $k \geq 1$, $k \in \mathbf{N}$, we have $a_{2k} = 1 > x$, so x cannot be an upper bound for $(a_n)_{n=2k}^{\infty}$, for any $k \in \mathbf{N}$, $k \geq 1$. We conclude that, for any $n \geq 1$, we have $\sup_{m\geq n} a_m = 1$. Therefore, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \lim_{n\to\infty} 1 = 1$. Now, note that $a_n \geq -1$ for all $n \geq 1$, so -1 is always a lower bound for $(a_n)_{n=1}^{\infty}$. However, if $x > -1$, then for any $k \geq 1$, $k \in \mathbf{N}$, we have $a_{2k+1} = -1 < x$, so x cannot be a lower bound for $(a_n)_{n=2k+1}^{\infty}$, for any $k \in \mathbf{N}$, $k \geq 1$. We conclude that, for any $n \geq 1$, we have $\inf_{m\geq n} a_m = -1$. Therefore, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \lim_{n\to\infty} -1 = -1$.

3. Question 3

Determine which of the following series converges. Justify your answer. (You are not allowed to compute any integrals or do integral comparison, since we have not covered that material yet.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution. From the dyadic criterion with $a_n = n^{-2}$, we have $a_{2^n} = 2^{-2n}$, so $2^n a_{2^n} = 2^n 2^{-2n} = 2^{-n}$, so $\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$, so the original series converges.

[1] March 25, 2025,

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

Solution. This series converges by the Alternating Series Test. The sequence is decreasing (since $n > m$ implies $n^{1/3} > m^{1/3}$, i.e. $1/n^{1/3} < 1/m^{1/3}$, for any m, n positive integers), so this test applies.

4. Question 4

Prove the Bolzano-Weierstrass Theorem. That is, prove the following:

Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence. That is, there exists a real number M such that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Then there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges.

Solution. Let $L := \limsup_{n\to\infty} a_n$. From the Comparison Principle, $|L| \leq M$. In particular, L is a real number. So, by a proposition in the notes (2.5.7(v)), L is a limit point of $(a_n)_{n=0}^{\infty}$. By another proposition in the notes (2.8.7), there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.

5. Question 5

(a) Find a function $f\colon \mathbf{R} \to \mathbf{R}$ such that, for every $x \in \mathbf{R}$, f is not continuous at x.

Solution. Define f so that $f(x) = 0$ when x is rational and $f(x) = 1$ when x is irrational. For any $x \in \mathbf{R}$, density of the rationals (and irrationals) implies that, for any $\varepsilon > 0$, the set $(x - \varepsilon, x + \varepsilon)$ always has a rational number and an irrational number. So, by definition of f, for any $\varepsilon > 0$, f takes both value 0 and value 1 in the set $(x - \varepsilon, x + \varepsilon)$. Consequently, $\lim_{y\to x} f(y)$ does not exist, for every $x \in \mathbf{R}$. Therefore, f is not continuous at any real value.

(b) Find a function $f\colon [0, \infty) \to \mathbf{R}$ which is continuous and bounded, where f attains its maximum somewhere, but f does not attain its minimum anywhere. That is, there is some $x \geq 0$ such that $f(x) \geq f(y)$ for all $y \geq 0$, but there is no $z \geq 0$ such that $f(z) \leq f(y)$ for all $y \geq 0$.

Solution. Let $f(x) = 1/(x+1)$ for all $x \geq 0$. Evidently f is decreasing, so $f(0) \geq f(x)$ for all $x \geq 0$. Also, f is positive so $f(x) > 0$ for all $x \geq 0$. But since f is strictly decreasing, for any $x \geq 0$, there exists $y > x$ such that $f(x) > f(y)$. That is, f does not attain a minimum value.