Math 425, Spring 2025, USC Instructor: Steven Heilman

Name: ____________________ USC ID: ________________ Date: ____________

Signature: ____________________. Discussion Section: ____________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	8	
2	8	
3	10	
4	10	
5	10	
Total:	46	

Reference Sheet

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges to** L if and only if, for every real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=0}^{\infty}$ is a **Cauchy sequence** if and only if, for any real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is **bounded** if and only if there exists $M \in \mathbf{R}$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Define the **limit superior** of the sequence to be

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{m\geq n} a_m.$$

Define the **limit inferior** of the sequence to be

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf_{m\geq n} a_m.$$

Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, define the N^{th} **partial sum** S_N of this series by $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit $L \in \mathbf{R}$ as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent**, and this infinite series **converges to** L.

The Root Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$. (i) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, the series $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. (iii) If $\alpha = 1$, no conclusion is asserted.

The Ratio Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers. (So, a_{n+1}/a_n is defined for any $n \geq m$.) (i) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. In particular, $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent.

Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is **continuous** at x_0 if and only if

$$\lim_{x\to x_0; x\in X} f(x) = f(x_0).$$

That is, the limit of f at x_0 in X exists, and this limit is equal to $f(x_0)$. We say that f is **continuous on** X (or we just say that f is **continuous**) if and only if f is continuous at x_0 for every $x_0 \in X$.

Intermediate Value Theorem. Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be function that is continuous on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.

1. Label the following statements as TRUE or FALSE.
If the statement is true, **EXPLAIN YOUR REASONING**.
If the statement is false, **PROVIDE A COUNTEREXAMPLE AND EXPLAIN YOUR REASONING**.

(a) (2 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

TRUE FALSE (circle one)

(b) (2 points) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ converges.

TRUE FALSE (circle one)

(c) (2 points) Let $f, g\colon \mathbf{R} \to \mathbf{R}$. If $\lim_{x\to 0}(f(x) + g(x))$ exists, then $\lim_{x\to 0} f(x)$ exists, and $\lim_{x\to 0} g(x))$ exists, and also
$\lim_{x\to 0}(f(x) + g(x)) = (\lim_{x\to 0} f(x)) + (\lim_{x\to 0} g(x))$.

TRUE FALSE (circle one)

(d) (2 points) If $\sum_{n=1}^{\infty} a_n$ converges, then any rearrangement of the sum also converges. That is, for any bijection $g\colon \mathbf{N} \to \mathbf{N}$, $\sum_{n=1}^{\infty} a_{g(n)}$ converges

TRUE FALSE (circle one)

2. [these examples were done in class]

   (a) (4 points) Let $a_n = 1/n$ for all positive integers n.
   Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. JUSTIFY YOUR ANSWER.

   (b) (4 points) Let $b_n = (-1)^n$ for all positive integers n.
   Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. JUSTIFY YOUR ANSWER.

3. Determine which of the following series converges. JUSTIFY YOUR ANSWER. (You are not allowed to compute any integrals or do integral comparison, since we have not covered that material yet.)

   [variants of these examples were done in class]

   (a) (5 points) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (b) (5 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

4. (10 points) Prove the Bolzano-Weierstrass Theorem. That is, prove the following:

   Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence. That is, there exists a real number M such that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Then there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges.

   [we proved this in class]

5. Justify your answers below.

[we went over examples like this in class]

(a) (5 points) Find a function $f\colon \mathbf{R} \to \mathbf{R}$ such that, for every $x \in \mathbf{R}$, f is not continuous at x. JUSTIFY YOUR ANSWER.

(b) (5 points) Find a function $f\colon [0, \infty) \to \mathbf{R}$ which is continuous and bounded, where f attains its maximum somewhere, but f does not attain its minimum anywhere. That is, there is some $x \geq 0$ such that $f(x) \geq f(y)$ for all $y \geq 0$, but there is no $z \geq 0$ such that $f(z) \leq f(y)$ for all $y \geq 0$. JUSTIFY YOUR ANSWER.

(Scratch paper)