425 Midterm 1 Solutions[1]

1. Question 1

(a) The set of rational numbers $\mathbf{Q}$ is countable.
TRUE. We proved this in class, Corollary 2.1.24
(b) The set of real numbers $\mathbf{R}$ is uncountable.
TRUE. We proved this in class, Corollary 2.1.30
(c) There is a set of cardinality larger than the real numbers. That is, there is an uncountable set that does not have the same cardinality as the real numbers.
TRUE. By Proposition 2.1.28, $2^{\mathbf{R}}$ is an uncountable set with cardinality different than $\mathbf{R}$. (Since $\mathbf{R} \subseteq 2^{\mathbf{R}}$, and since $\mathbf{R}$ is uncountable, $2^{\mathbf{R}}$ is uncountable.)
(d) The set $\mathbf{N} \times \mathbf{N} = \{(a, b) \colon a \in \mathbf{N}, b \in \mathbf{N}\}$ is uncountable.
FALSE. We showed in class that $\mathbf{N} \times \mathbf{N}$ is countable in Lemma 2.1.22.

2. Question 2

Prove the following:
For any positive integer n,

$$2^{n+1} > n^2.$$

Solution. We prove this by induction on n. The base case $n = 1$ follows since it says $2^2 > 1^2$, i.e. $4 > 1$, which is true. Also note the base case $n = 2$ holds, since it says $2^3 > 2^2$, i.e. $8 > 4$, which is true. We then do the inductive step. Assume the assertion holds for $n \geq 1$, and we are required to show it holds in the case $n+1$. Using the inductive hypothesis, we have

$$2^{n+2} = 2 \cdot 2^{n+1} > 2n^2.$$

Also $(n+1)^2 = n^2+2n+1$, so it remains to show that $2n^2 \geq n^2+2n+1$, i.e. that $n^2 \geq 2n+1$. In the case $n \geq 3$, we have $n^2 \geq 3n = 2n + n \geq 2n + 1$. So we are done the inductive step. The desired assertion then follows by induction (using $n = 2$ as the base case.)

3. Question 3

Prove the reverse triangle inequality. That is, show:
For any rational numbers x, y, we have

$$|x - y| \geq \Big| |x| - |y| \Big|.$$

(Hint: you can freely use the usual triangle inequality.)
Solution. Using the usual triangle inequality, we have

$$|x| = |(x - y) + y| \leq |x - y| + |y| \quad \text{and} \quad |y| = |(y - x) + x| \leq |y - x| + |x|.$$

Thus it follows that

$$|x| - |y| \leq |x - y| \quad \text{and} \quad |y| - |x| \leq |y - x| = |x - y|. \qquad (*)$$

By definition of the absolute value, $||x| - |y|| = |x| - |y|$ or $||x| - |y|| = -(|x| - |y|)$. In the first case, the first part of $(*)$ concludes the proof, and in the second case, the second part of

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$(*)$ concludes the proof. So regardless of whether $||x|-|y|| = |x|-|y|$ or $||x|-|y|| = |y|-|x|$, it follows that $||x|-|y|| \leq |x-y|$ as desired.

4. Question 4

Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers that converges to a real number x. Let $(b_n)_{n=0}^{\infty}$ be a sequence of rational numbers that converges to a real number y.

Show that the sequence $(a_n+b_n)_{n=0}^{\infty}$ converges to the real number $x+y$.

Solution. We have from the triangle inequality that for all $n \geq 0$,

$$|a_n + b_n - (x+y)| \leq |a_n - x| + |b_n - y| . \qquad (*)$$

Let $\varepsilon > 0$. Then there exists $N_1, N_2 > 0$ such that $|a_n - x| < \varepsilon/2$ for all $n \geq N_1$, and $|b_n - x| < \varepsilon/2$ for all $n \geq N_2$. Define then $N := \max(N_1.N_2)$. Then for all $n \geq N$, we have from $(*)$ that

$$|a_n + b_n - (x+y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, $(a_n+b_n)_{n=0}^{\infty}$ converges to the real number $x+y$.

5. Question 5

Let x be a rational number. Prove that there exists a unique integer n such that $n \leq x < n+1$. In particular, there exists an integer N such that $x < N$.

Solution. Let $x \in \mathbb{Q}$ and write it as a quotient $x = \frac{p}{q}$ of two integers $p, q \in \mathbb{Z}$ with $q > 0$. Assume for now that $p \geq 0$. Then by the Euclidean algorithm, there exists $m, r \in \mathbf{N}$ with $0 \leq r < q$ such that $p = mq + r$. This gives

$$x = \frac{p}{q} = \frac{mq+r}{q} = m + \frac{r}{q}.$$

But since $0 \leq \frac{r}{q} < 1$, it follows that $m \leq m + r/q \leq m+1$. Since $x = m + r/q$, we get $m \leq x < m+1$ and this proves the existence. In case $p < 0$, we apply the above reasoning to $-p/q$ to get $m \leq -x \leq m+1$, so that $-(m+1) \leq x \leq -m$.

To prove the uniqueness, let $m, n \in \mathbb{Z}$ satisfy $m < x < m+1$ and $n < x < n+1$, respectively. We claim that in fact $m = n$. By relabeling if required, we may assume that $m \leq n$. Then $n = m + a$ for some $a \in \mathbb{N}$. But if $a \neq 0$, then $a \geq 1$ and this implies

$$x < m + 1 \leq m + a = n \leq x$$

That is, $x < x$, a contradiction! Therefore $a = 0$ and hence $m = n$.