Math 425, Spring 2025, USC — Instructor: Steven Heilman

Name: ______________ USC ID: ______________ Date: ______________

Signature: ______________. Discussion Section: ______________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
Total:	48	

Do not write in the table to the right. Good luck![a]

Reference Sheet

The set of natural numbers is

$$\mathbf{N} := \{0, 1, 2, 3, \ldots\}.$$

The set of integers is

$$\mathbf{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

The set of rational number is

$$\mathbf{Q} := \left\{\frac{a}{b} : \quad a, b \in \mathbf{Z},\, b \neq 0\right\}.$$

The set $\mathbf{R}$ denotes the set of real numbers.

Let X and Y be sets. We say that a function $f\colon X \to Y$ is a **bijection** if every $y \in Y$ has exactly one $x \in X$ such that $f(x) = y$. Two sets X and Y have the same **cardinality** if there exists a bijection $f\colon X \to Y$.

A set X is **countable** if there exists a bijection $f\colon X \to \mathbf{N}$. A set X is **finite** if X is empty, or if there exists $n \in \mathbf{N}$ and there exists a bijection $f\colon X \to \{1, \ldots, n\}$. A set X is **infinite** if X is not finite. A set X is **at most countable** if X is finite or countable. A set X is **uncountable** if: X is not finite, and X is not countable.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges** to $x \in \mathbf{R}$ if: for all $\varepsilon \in \mathbf{R}$ with $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > 0$ such that, for all $n \geq N$, $|a_n - x| < \varepsilon$.

1. Label the following statements as TRUE or FALSE.
   If the statement is true, **EXPLAIN YOUR REASONING**.
   If the statement is false, **PROVIDE A COUNTEREXAMPLE AND EXPLAIN YOUR REASONING**.

   (a) (2 points) The set of rational numbers $\mathbf{Q}$ is countable.

   TRUE FALSE (circle one)

   (b) (2 points) The set of real numbers $\mathbf{R}$ is uncountable.

   TRUE FALSE (circle one)

   (c) (2 points) There is a set of cardinality larger than the real numbers. That is, there is an uncountable set that does not have the same cardinality as the real numbers.

   TRUE FALSE (circle one)

   (d) (2 points) The set $\mathbf{N} \times \mathbf{N} = \{(a, b) \colon a \in \mathbf{N}, b \in \mathbf{N}\}$ is uncountable.

   TRUE FALSE (circle one)

2. (10 points) Prove the following:

For any positive integer n,

$$2^{n+1} > n^2.$$

3. (10 points) Prove the reverse triangle inequality. That is, show:

   For any rational numbers x, y, we have

$$|x - y| \geq \Big| |x| - |y| \Big|.$$

   (Hint: you can freely use the usual triangle inequality.)

   [This was a repeated homework question.]

4. (10 points)

Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers that converges to a real number x.
Let $(b_n)_{n=0}^{\infty}$ be a sequence of rational numbers that converges to a real number y.

Show that the sequence $(a_n + b_n)_{n=0}^{\infty}$ converges to the real number $x + y$.

[This was a repeated homework question.]

5. (10 points)

   Let x be a rational number.
   Prove that there exists a **unique** integer n such that $n \leq x < n + 1$.
   (Hint: you can freely use the Euclidean Algorithm, which says: for any natural numbers p, q with $q > 0$, there exist natural numbers m, r with $0 \leq r < q$ such that $p = mq + r$.)

   [This was a repeated homework question.]

(Scratch paper)