Please provide complete and well-written solutions to the following exercises.

Due October 12, 12PM noon PST, to be uploaded as a single PDF document to Gradescope.

Homework 4

Exercise 1. Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

Let Y be the estimator $Y = 1_{\{X_1=0\}}$. Suppose we want to estimate $e^{-\lambda}$.

- Find a method of moments estimator for $e^{-\lambda}$. Is this estimator consistent?
- Show that Y is unbiased for $e^{-\lambda}$.
- Show that $\sum_{i=1}^n X_i$ is sufficient for $e^{-\lambda}$.
- Compute $W_n := \mathbf{E}_\lambda(Y \mid \sum_{i=1}^n X_i)$, as in the Rao-Blackwell Theorem.
- As $n \to \infty$, does W_n converge in any sense? If so, what does it converge to? Does this mean that $W_1, W_2, \ldots$ is consistent?

Exercise 2. Let $X_1, \ldots, X_n$ be a random sample of size n from the uniform distribution on $[0, \theta]$ where $\theta > 0$ is unknown.

On a previous homework, we showed that

$$X_{(n)} = \max_{1 \leq i \leq n} X_i$$

is a sufficient statistic for θ.

- Show that $2X_1$ is an unbiased estimator of θ.
- Compute $W := \mathbf{E}_\theta(2X_1 \mid X_{(n)})$, as in the Rao-Blackwell Theorem. (Hint: with probability $1/n$, $X_1 = X_{(n)}$. And with probability $1 - 1/n$, $X_1 < X_{(n)}$, and if additionally $X_{(n)} = x$, then X_1 is uniform on $(0, x)$.) Using whatever method you wish, show that W is unbiased for θ.
- A method of moments estimator for θ is $2\frac{1}{n}\sum_{i=1}^n X_i$. Compute

$$\mathbf{E}_\theta\Big(2\frac{1}{n}\sum_{i=1}^n X_i \,\Big|\, X_{(n)}\Big).$$

Exercise 3. Let $X_1, \ldots, X_n$ be a random sample of size n from the Bernoulli distribution with $0 < \theta < 1$ unknown. (So, $\mathbf{P}(X_1 = 1) = \theta$ and $\mathbf{P}(X_1 = 0) = 1 - \theta$.)

In class, we showed that $\sum_{i=1}^n X_i$ is consistent for θ, and also that

$$\mathbf{E}_\theta\Big(X_1 \,\Big|\, \sum_{i=1}^n X_i\Big) = \frac{1}{n}\sum_{i=1}^n X_i.$$

That is, the Rao-Blackwell Theorem suggests that the sample mean has small variance among all unbiased estimators for θ.

- Compute the Fisher information $I_{X_1}(\theta)$.
- Compute the Fisher information $I_{(X_1,\ldots,X_n)}(\theta)$.
- Show that $\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\theta(1-\theta)}{n}$.
- Does the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$ achieve equality in the Cramer-Rao inequality? If so, then $\frac{1}{n}\sum_{i=1}^{n} X_i$ is UMVU.