Please provide complete and well-written solutions to the following exercises.

Due April 20, 2PM PST, to be uploaded in blackboard as a single PDF document (in the Assignments tab).

Homework 11

Exercise 1. Compute the characteristic function of a uniformly distributed random variable on $[-1,1]$. (Some of the following formulas might help to simplify your answer: $e^{it} = \cos(t) + i\sin(t)$, $\cos(t) = [e^{it} + e^{-it}]/2$, $\sin(t) = [e^{it} - e^{-it}]/[2i]$, $t \in \mathbf{R}$.)

Exercise 2. Let X be a random variable. Assume we can differentiate under the expected value of $\mathbf{E}e^{itX}$ any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}|_{t=0}\phi_X(t) = i^n \mathbf{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the characteristic function.

Exercise 3. Let X be a random variable such that $\mathbf{E}\left|X\right|^3 < \infty$. Prove that for any $t \in \mathbf{R}$,

$$\mathbf{E}e^{itX} = 1 + it\mathbf{E}X - t^2\mathbf{E}X^2/2 + o(t^2).$$

That is,

$$\lim_{t\to 0} t^{-2}\left|\mathbf{E}e^{itX} - [1 + it\mathbf{E}X - t^2\mathbf{E}X^2/2]\right| = 0$$

(Hint: it may be helpful to use Jensen's inequality, to first justify that $\mathbf{E}\left|X\right| < \infty$ and $\mathbf{E}X^2 < \infty$. Then, use the Taylor expansion with error bound: $e^{iy} = 1 + iy - y^2/2 - (i/2)\int_0^y (y-s)^2 e^{is} ds$, which is valid for any $y \in \mathbf{R}$.)

Actually, this same bound holds only assuming $\mathbf{E}X^2 < \infty$, but the proof of that bound requires things we have not discussed.

Exercise 4. Let X, Y, Z be independent and uniformly distributed on $[0,1]$. Note that f_X is not a continuous function.

Using convolution, compute f_{X+Y}. Draw f_{X+Y}. Note that f_{X+Y} is a continuous function, but it is not differentiable at some points.

Using convolution, compute f_{X+Y+Z}. Draw f_{X+Y+Z}. Note that f_{X+Y+Z} is a differentiable function, but it does not have a second derivative at some points.

Make a conjecture about how many derivatives $f_{X_1+\cdots+X_n}$ has, where $X_1, \ldots, X_n$ are independent and uniformly distributed on $[0,1]$. You do not have to prove this conjecture. The idea of this exercise is that convolution is a kind of average of functions. And the more averaging you do, the more derivatives $f_{X_1+\cdots+X_n}$ has.

Exercise 5. Construct two random variables X, Y such that X and Y are each uniformly distributed on $[0,1]$, and such that $\mathbf{P}(X+Y=1)=1$.

Then construct two random variables W, Z such that W and Z are each uniformly distributed on $[0,1]$, and such that $W+Z$ is uniformly distributed on $[0,2]$.

(Hint: there is a way to do each of the above problems with about one line of work. That is, there is a way to solve each problem without working very hard.)

Exercise 6. Let X be a standard Gaussian random variable. Let $t>0$ and let n be a positive even integer. Show that

$$\mathbf{P}(X>t) \leq \frac{(n-1)(n-3)\cdots(3)(1)}{t^n}.$$

That is, the function $t \mapsto \mathbf{P}(X>t)$ decays faster than any monomial.

Exercise 7. Let X be a random variable. Let $t>0$. Show that

$$\mathbf{P}(|X|>t) \leq \frac{\mathbf{E}X^4}{t^4}.$$

Exercise 8 (The Chernoff Bound)**.** Let X be a random variable and let $r>0$. Show that, for any $t>0$,

$$\mathbf{P}(X>r) \leq e^{-tr} M_X(t).$$

Consequently, if $X_1, \ldots, X_n$ are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) \leq e^{-trn}(M_{X_1}(t))^n.$$

For example, if $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $0<p<1$, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{X_1+\cdots+X_n}{n} - p > r\right) \leq e^{-trn}(e^{-tp}[pe^t+(1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbf{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i-p)\right|>r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n}\sum_{i=1}^{n} X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbf{P}\left(\left|\frac{X_1+\cdots+X_n}{n} - p\right| > r\right) \leq \frac{2p(1-p)}{r}, \quad \mathbf{P}\left(\left|\frac{X_1+\cdots+X_n}{n} - p\right| > r\right) \leq \frac{p(1-p)}{nr^2}.$$