Please provide complete and well-written solutions to the following exercises.

Due April 13, 2PM PST, to be uploaded in blackboard as a single PDF document (in the Assignments tab).

Homework 10

Exercise 1. Let X be a binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 2$ and $\mathbf{E}Y^2 = 6$.

Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

Let W be a discrete random variable such that $\mathbf{P}(W = 0) = 1/2$ and $\mathbf{P}(W = 4) = 1/2$, so that $\mathbf{E}W = 2$ and $\mathbf{E}W^2 = 8$.

Assume that X, Y, Z and W are all independent. Compute

$$\mathrm{var}(X + Y + Z + W).$$

Exercise 2. Let $X_1, \ldots, X_n$ be random variables with finite variance. Define an $n \times n$ matrix A such that $A_{ij} = \mathrm{cov}(X_i, X_j)$ for any $1 \leq i, j \leq n$. Show that the matrix A is positive semidefinite. That is, show that for any $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we have

$$y^T A y = \sum_{i,j=1}^{n} y_i y_j A_{ij} \geq 0.$$

Exercise 3. Using the definition of convergence, show that the sequence of numbers $1, 1/2, 1/3, 1/4, \ldots$ converges to 0.

Exercise 4 (Uniqueness of limits)**.** Let $x_1, x_2, \ldots$ be a sequence of real numbers. Let $x, y \in \mathbf{R}$. Assume that $x_1, x_2, \ldots$ converges to x. Assume also that $x_1, x_2, \ldots$ converges to y. Prove that $x = y$. That is, a sequence of real numbers cannot converge to two different real numbers.

Exercise 5. Let X be a random variable. Assume that $M_X(t)$ exists for all $t \in \mathbf{R}$, and assume we can differentiate under the expected value any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}|_{t=0} M_X(t) = \mathbf{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the moment generating function.

Exercise 6. Let X be a standard Gaussian random variable. Compute an explicit formula for the moment generating function of X. (Hint: completing the square might be helpful.) From this explicit formula, compute an explicit formula for all moments of the Gaussian random variable. (The $2n^{th}$ moment of X should be something resembling a factorial.)

Exercise 7. Construct two random variables $X, Y : \Omega \to \mathbf{R}$ such that $X \neq Y$ but $M_X(t), M_Y(t)$ exist for all $t \in \mathbf{R}$, and such that $M_X(t) = M_Y(t)$ for all $t \in \mathbf{R}$.

Exercise 8. Unfortunately, there exist random variables X, Y such that $\mathbf{E}X^n = \mathbf{E}Y^n$ for all $n = 1, 2, 3, \ldots$, but such that X, Y do not have the same CDF. First, explain why this does not contradict the Lévy Continuity Theorem, Weak Form. Now, let $-1 < a < 1$, and define a density

$$f_a(x) := \begin{cases} \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x)^2}{2}} (1 + a \sin(2\pi \log x)) & \text{, if } x > 0 \\ 0 & \text{, otherwise.} \end{cases}$$

Suppose X_a has density f_a. If $-1 < a, b < 1$, show that $\mathbf{E}X_a^n = \mathbf{E}X_b^n$ for all $n = 1, 2, 3, \ldots$. (Hint: write out the integrals, and make a change of variables $s = \log(x) - n$.)