407 Final v3 Solutions[1]

1. Question 1

Let X be a random variable uniformly distributed in $[0,1]$.
(That is, X has PDF $f_X(x) = 1$ when $x \in [0,1]$, and $f_X(x) = 0$ when $x \notin [0,1]$.)
Let Y be a random variable uniformly distributed in $[0,1]$.
Assume that X and Y are independent.

- Compute $\mathbf{P}(X > 3/4)$.
- Compute $\mathbf{E}X$.
- Compute $\mathbf{P}(X + Y \leq 1/2)$.

In all cases, simplify your answer to the best of your ability.

Solution. Since X has PDF $f_X = 1_{[0,1]}$, we have $\mathbf{P}(X > 3/4) = \int_{3/4}^1 dx = 1/4$ and $\mathbf{E}X = \int_0^1 x dx = 1/2$. Since X and Y are independent, they have joint PDF $f_{X,Y} = 1_{[0,1]^2}$, so that

$$\begin{aligned}\mathbf{P}(X + Y \leq 1/2) &= \int_{\{(x,y)\in\mathbf{R}^2\colon\, x\geq 0, y\geq 0, x+y\leq 1/2\}} = \int_{x=0}^{x=1/2}\int_{y=0}^{y=1/2-x} dydx\\ &= \int_{x=0}^{x=1/2} (1/2 - x)dx = ((1/2)x - x^2/2)_{x=0}^{x=1/2}\\ &= (1/2)^2 - (1/2)^3 = 1/4 - 1/8 = 1/8.\end{aligned}$$

2. Question 2

Let X and Y be independent random variables. Assume that X is uniformly distributed in $[-1,1]$, and Y is uniformly distributed in $[-1,1]$.

- Compute $\mathbf{E}(X^2Y)$.
- Compute $\mathbf{P}(X^2 + Y^2 \geq 1)$.

In all cases, simplify your answer to the best of your ability.

Simplify your answer to the best of your ability.

Solution. Since X, Y are independent, $\mathbf{E}(X^2Y) = \mathbf{E}X^2\mathbf{E}Y = 0$, since $\mathbf{E}Y = 0$. (We have $\mathbf{E}Y = \int_{-1}^1 ydy/2 = 0$.)

Since X, Y are independent, we have $f_{X,Y}(x,y) = f_X(x)f_Y(y) = (1/2)^2 = (1/4)$ if $x, y \in [-1,1]$ and 0 otherwise. So, by definition of joint PDF,

$$\begin{aligned}\mathbf{P}(X^2 + Y^2 \geq 1) &= 1 - \mathbf{P}(X^2 + Y^2 < 1) = 1 - \int_{\{(x,y)\in\mathbf{R}^2\colon\, x^2+y^2<1\}} f_{X,Y}(x,y)dxdy\\ &= 1 - \frac{1}{4}\int_{\{(x,y)\in\mathbf{R}^2\colon\, x^2+y^2<1\}} dxdy = 1 - \frac{1}{4}\pi = 1 - \pi/4.\end{aligned}$$

In the last line, we used that the area of the unit disc $\{(x,y) \in \mathbf{R}^2 \colon x^2 + y^2 < 1\}$ is π.

3. Question 3

Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs twenty times, so that twenty spheres have been placed in bins. (All of the sphere placements up to this point are independent of each other).

Suppose you now flip a fair coin. (A fair coin has probability 1/2 of landing heads, and probability 1/2 of landing tails). (The coin flip result is independent of all of the sphere placements.) If the coin lands heads, you then place another ten spheres randomly into the bins (with each sphere being equally likely to appear in any of the five bins).

What is the expected number of empty bins?

Simplify your answer to the best of your ability. (As usual, show your work.)

Solution. Let A be the event that the coin flip is heads. Let N be the number of empty bins. We are required to compute $\mathbf{E}N$. From the Total Expectation Theorem,

$$\mathbf{E}N = \mathbf{E}(N|A)\mathbf{P}(A) + \mathbf{E}(N|A^c)\mathbf{P}(A^c).$$

By its definition, $\mathbf{P}(A) = \mathbf{P}(A^c) = 1/2$, so

$$\mathbf{E}N = (1/2)[\mathbf{E}(N|A) + \mathbf{E}(N|A^c)].$$

For each $1 \leq i \leq 5$, let X_i be 1 if bin i is empty, and $X_i = 0$ otherwise. Then $N = \sum_{i=1}^5 X_i$ and $\mathbf{E}(N|A) = \sum_{i=1}^5 \mathbf{E}(X_i|A)$, $\mathbf{E}(N|A^c) = \sum_{i=1}^5 \mathbf{E}(X_i|A^c)$. Since X_i only takes values 0 or 1, we then have

$$\mathbf{E}(N|A) = \sum_{i=1}^5 \mathbf{P}(X_i = 1|A), \qquad \mathbf{E}(N|A^c) = \sum_{i=1}^5 \mathbf{P}(X_i = 1|A^c)$$

If A occurs, there are thirty total spheres placed in the bins, with all placements being independent and uniformly random. Since all sphere placements are equally likely, $\mathbf{P}(X_i = 1|A)$ is the probability that all thirty spheres lie in the other bins (other than the i^{th} bin), i.e. this probability is $(4/5)^{30}$. Similarly, if A^c occurs, there are twenty total spheres placed in the bins, with all placements being independent and uniformly random. Since all sphere placements are equally likely, $\mathbf{P}(X_i = 1|A^c)$ is the probability that all twenty spheres lie in the other bins, i.e. this probability is $(4/5)^{20}$. Combining the above, we have

$$\mathbf{E}(N|A) = \sum_{i=1}^5 (4/5)^{30} = 5(4/5)^{30}, \qquad \mathbf{E}(N|A^c) = \sum_{i=1}^5 (4/5)^{20} = 5(4/5)^{20}.$$

$$\mathbf{E}N = (1/2)[\mathbf{E}(N|A) + \mathbf{E}(N|A^c)] = (1/2)[5(4/5)^{30} + 5(4/5)^{20}].$$

4. Question 4

Let X, Y be independent standard Gaussian random variables. (That is, X has PDF $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $\forall\, x \in \mathbf{R}$.)

Let $Z := X/|Y|$.

Find the PDF of Z. (Justify your answer.) (Simplify your answer to the best of your ability.) (PDF is an acronym for: Probability Density Function.)

Solution. For any $t \in \mathbf{R}$, let $A_t := \{(x,y) \in \mathbf{R}^2 \colon x \leq t\,|y|\}$. Then, using polar coordinates, if $t \geq 0$ we have

$$\begin{aligned}
\mathbf{P}(Z \leq t) = \mathbf{P}(X \leq t\,|Y|) = \mathbf{P}((X,Y) \in A_t) &= \frac{1}{2\pi}\int_{A_t} e^{-(x^2+y^2)/2}dxdy \\
&= \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=t|y|} e^{-(x^2+y^2)/2}dxdy \\
&= \frac{1}{2\pi}\int_{\theta=\tan^{-1}(1/t)}^{\theta=2\pi-\tan^{-1}(1/t)}\int_{r=0}^{r=\infty} re^{-r^2/2}drd\theta \\
&= \frac{1}{2\pi}\int_{\theta=\tan^{-1}(1/t)}^{\theta=2\pi-\tan^{-1}(1/t)} d\theta = 1 - \frac{1}{\pi}\tan^{-1}(1/t).
\end{aligned}$$

So, from the Chain rule, if $t > 0$, then

$$f_Z(t) = \frac{d}{dt}\Big(1 - \frac{1}{\pi}\tan^{-1}(1/t).\Big) = -\frac{1}{\pi}\frac{1}{1+(1/t)^2}\cdot(-t^{-2}) = \frac{1}{\pi(t^2+1)}, \qquad \forall\, t > 0.$$

Similarly, if $t < 0$, then $\mathbf{P}(Z \leq t) = \frac{1}{\pi}\tan^{-1}(1/\,|t|)$, and

$$f_Z(t) = \frac{d}{dt}\Big(\frac{1}{\pi}\tan^{-1}(-1/t).\Big) = \frac{1}{\pi}\frac{1}{1+(1/t)^2}\cdot(t^{-2}) = \frac{1}{\pi(t^2+1)}, \qquad \forall\, t < 0.$$

In conclusion,

$$f_Z(t) = \frac{1}{\pi(t^2+1)}, \qquad \forall\, t \in \mathbf{R}.$$

5. Question 5

Let $X_1, X_2, \ldots$ be independent random variables, each with exponential distribution with parameter $\lambda = 1$. (That is, X_1 has PDF $f_X(x) = e^{-x}$ when $x \geq 0$, and $f_X(x) = 0$ for $x < 0$.)

For any $n \geq 1$, let $Y_n := \max(X_1, \ldots, X_n)$. Let $0 < a < 1 < b$.

- Show that $\lim_{n\to\infty}\mathbf{P}(Y_n \leq a\log n) = 0$ and $\lim_{n\to\infty}\mathbf{P}(Y_n \leq b\log n) = 1$.
- Conclude that, for all $\varepsilon > 0$, $\lim_{n\to\infty}\mathbf{P}(|Y_n/\log n - 1| > \varepsilon) = 0$

Solution. Since $X_1, \ldots, X_n$ are i.i.d.

$$\mathbf{P}(Y_n \leq t) = \mathbf{P}(\max(X_1, \ldots, X_n) \leq t) = \mathbf{P}(X_1 \leq t)^n = [\int_0^t e^{-x}dx] = [1 - e^{-t}]^n.$$

So, choosing $t = a\log n$, we have

$$\mathbf{P}(Y_n \leq t) = [1 - e^{-a\log n}]^n = [1 - n^{-a}]^n.$$

Taking logs, and using the power series expansion $\log(1-x) = -x + o(x)$ as $x \to 0$,

$$\log \mathbf{P}(Y_n \leq a\log n) = n\log(1 - n^{-a}) = n(n^{-a} + o(n^{-a})) = n^{1-a} + o(n^{1-a})$$

So, if $a < 1$, this quantity goes to ∞ as $n \to \infty$, and if $a > 1$, this quantity goes to zero as $n \to \infty$. We conclude that $\lim_{n\to\infty}\mathbf{P}(Y_n \leq a\log n) = \lim_{n\to\infty} e^{n^{1-a}} = \infty$ and $\lim_{n\to\infty}\mathbf{P}(Y_n \leq b\log n) = \lim_{n\to\infty} e^{n^{1-b}} = e^0 = 1$.

Finally, let $\varepsilon > 0$. Then, using $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$,

$$\begin{aligned}\mathbf{P}(|Y_n/\log n - 1| > \varepsilon) &= \mathbf{P}(Y_n/\log n < 1 - \varepsilon \text{ or } Y_n/\log n > 1 + \varepsilon)\\ &\leq \mathbf{P}(Y_n/\log n < 1 - \varepsilon) + \mathbf{P}(Y_n/\log n > 1 + \varepsilon)\\ &= \mathbf{P}(Y_n/\log n < 1 - \varepsilon) + 1 - \mathbf{P}(Y_n/\log n \leq 1 + \varepsilon).\end{aligned}$$

From what we already did, the first probability goes to zero as $n \to \infty$, and the last probability goes to 1 as $n \to \infty$. We therefore conclude that

$$\lim_{n\to\infty} \mathbf{P}(|Y_n/\log n - 1| > \varepsilon) = 0$$

6. Question 6

Suppose you flip a fair coin 120 times. During each coin flip, this coin has probability 1/2 of landing heads, and probability 1/2 of landing tails.

Let A be the event that you get more than 90 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{60}.$$

Solution 1. For any $n \geq 1$, define X_n so that

$$X_n = \begin{cases} 1 & \text{, if the } n^{th} \text{ coin flip is heads} \\ 0 & \text{, if the } n^{th} \text{ coin flip is tails.} \end{cases}$$

By its definition $\mathbf{E}X_n = 1/2$ and $\mathrm{var}(X_n) = (1/2)(1/4) + (1/2)(1/4) = 1/4$.

Let $S := X_1 + \cdots + X_{120}$ be the number of heads that are flipped. Then $\mathbf{E}S = 60$, and $\mathrm{var}(S) = 120\mathrm{var}(X_1) = 30$. Markov's inequality says, for any $t > 0$

$$\mathbf{P}(S > t) \leq \mathbf{E}S/t = 60/t.$$

This is not helpful. Instead, we use Chebyshev's inequality. This says, for any $t > 0$,

$$\mathbf{P}(|S - 60| > t) \leq t^{-2}\mathrm{var}(S) = 30t^{-2}.$$

Choosing $t = 30$ shows that $\mathbf{P}(|S - 60| > 10) \leq 1/30$. Now, using symmetry of S (interchanging the roles of heads and tails),

$$\mathbf{P}(|S - 60| > 30) = \mathbf{P}(S < 30) + \mathbf{P}(S > 90) = 2\mathbf{P}(S > 90).$$

So,

$$2\mathbf{P}(S > 90) = \mathbf{P}(|S - 60| > 30) \leq 1/30.$$

Solution 2. We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Chernoff bound. Since S is a sum of 120 independent identically distributed random variables, Proposition 2.43 from the notes says

$$M_S(t) = (M_{X_1}(t))^{120}, \qquad \forall\, t \in \mathbf{R}.$$

So, the Chernoff bound says, for any $r, t > 0$,

$$\mathbf{P}(S > r) \leq e^{-tr}(M_{X_1}(t))^{120} = e^{-tr}((1/2)(1 + e^t))^{120} \qquad (*).$$

Setting $f(t) = e^{-rt}(1 + e^t)^{120}$ and solving $f'(t) = 0$ for t shows that $t = \log(3)$ minimizes the quantity $f(t)$. So, choosing $r = 90$ and $t = \log(3)$ in $(*)$ gives

$$\mathbf{P}(S > 90) \leq e^{-tr}((1/2)(1 + 4))^{120} = (3)^{-90}(5/2)^{120} \leq .0006 < 1/60.$$

Solution 3. (The following solution based on the Central Limit Theorem only received partial credit, since it only approximately shows that $\mathbf{P}(A) < 1/10$.) We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Central Limit Theorem. Since $X_1, X_2, \ldots$ are independent identically distributed random variables with mean 1/2 and variance 1/4, the Central Limit Theorem implies that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n/2}{\sqrt{(1/4)}\sqrt{n}} > t\right) = \int_t^\infty e^{-x^2/2} dx/\sqrt{2\pi}.$$

So, choosing $n = 120$ and $t = \sqrt{30}$, we have the approximation

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_{120} - 60}{\sqrt{(1/4)}\sqrt{120}} > \sqrt{30}\right) \approx \int_{\sqrt{30}}^\infty e^{-x^2/2} dx/\sqrt{2\pi}.$$

Simplifying a bit,

$$\mathbf{P}\left(S - 60 > 30\right) \approx \int_{\sqrt{30}}^\infty e^{-x^2/2} dx/\sqrt{2\pi}.$$

Using $\sqrt{30} > 3$ and the approximation $\int_3^\infty e^{-x^2/2} dx/\sqrt{2\pi} \approx .0014$, we have

$$\mathbf{P}(S > 50) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2} dx/\sqrt{2\pi} \leq \int_2^\infty e^{-x^2/2} dx/\sqrt{2\pi} \approx .0014 < 1/60.$$

7. Question 7

Let $X_1, X_2, \ldots$ be i.i.d. (independent identically distributed) random variables. Assume that $\mathbf{E}\,|X_1| < \infty$ and $\mathrm{Var}(X_1) = 0$. Denote $\mu := \mathbf{E}X_1$.

Does the random variable

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}}$$

converge in distribution to some random variable Z as $n \to \infty$? If so, what is the CDF of Z? (Here CDF denotes cumulative distribution function.)

(Justify your answer.)

Solution. Since $\mathrm{Var}(X_1) = 0$, X_1 is constant. Since $X_1, X_2, \ldots$ are i.i.d., all of these random variables are equal to the same constant, so that $X_1 + \cdots + X_n - n\mathbf{E}X_1 = 0$. So, for any $n \geq 1$,

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}} = 0.$$

So, as $n \to \infty$, this random variable converges in distribution to the random variable $Z = 0$. Z has CDF given by $\mathbf{P}(Z \leq t) = 0$ if $t < 0$ and $\mathbf{P}(Z \leq t) = 1$ if $t \geq 1$.

8. Question 8

Let $X_1, X_2, \ldots$ be i.i.d. (independent identically distributed) random variables. Fix a real number $0 < \alpha \leq 2$. Assume X_1 has a characteristic function given by

$$\phi_{X_1}(t) = \mathbf{E}^{\sqrt{-1}\,tX_1} = e^{-|t|^\alpha}, \qquad \forall\, t \in \mathbf{R}$$

- Prove that

$$\phi_{\left(\frac{X_1+\cdots+X_n}{n^{1/\alpha}}\right)}(t) = \phi_{X_1}(t), \qquad \forall\, t \in \mathbf{R}.$$

- For what values of $0 < \alpha \leq 2$ does the random variable

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

converge in distribution to a Gaussian random variable with positive variance, as $n \to \infty$? (Justify your answer.)

Solution. Since $X_1, \ldots, X_n$ are i.i.d. we have

$$\begin{aligned}\phi_{\left(\frac{X_1+\cdots+X_n}{n^{1/\alpha}}\right)}(t) &= \prod_{i=1}^{n} \phi_{X_i/n^{1/\alpha}}(t) = \prod_{i=1}^{n} \mathbf{E} e^{itX_1/n^{1/\alpha}} = \prod_{i=1}^{n} \phi_{X_1}(t/n^{1/\alpha}) \\ &= \left(e^{-\left|t/n^{1/\alpha}\right|^{\alpha}}\right)^{n} = \left(e^{-|t|^{\alpha}/n}\right)^{n} = e^{-|t|^{\alpha}} = \phi_{X_1}(t).\end{aligned}$$

Since $\frac{X_1+\cdots+X_n}{n^{1/\alpha}}$ has the same characteristic function as X_1, we conclude from the Lévy Continuity Theorem that $\frac{X_1+\cdots+X_n}{n^{1/\alpha}}$ has the same CDF as X_1. So, we can write (using $0 < \alpha \leq 2$, so $1/\alpha \geq 1/2$, so $1/2 - 1/\alpha \leq 0$)

$$\frac{X_1 + \cdots + X_n}{n^{1/2}} = \frac{1}{n^{1/2-1/\alpha}} \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} = n^{1/\alpha - 1/2} \cdot \frac{X_1 + \cdots + X_n}{n^{1/\alpha}}$$

For any $t > 0$, we write

$$\begin{aligned}\mathbf{P}\Big(-t \leq \frac{X_1 + \cdots + X_n}{n^{1/2}} \leq t\Big) &= \mathbf{P}\Big(-t \leq n^{1/\alpha-1/2} \cdot \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \leq t\Big) \\ &= \mathbf{P}\Big(-tn^{1/2-1/\alpha} \leq \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \leq tn^{1/2-1/\alpha}\Big)\end{aligned}$$

If $\alpha < 2$, $1/\alpha - 1/2 > 0$. So, as $n \to \infty$, using Continuity of the Probability Law,

$$\lim_{n\to\infty} \mathbf{P}\Big(-t \leq \frac{X_1 + \cdots + X_n}{n^{1/2}} \leq t\Big) = \mathbf{P}\Big(\frac{X_1 + \cdots + X_n}{n^{1/\alpha}} = 0\Big)$$

That is, as $n \to \infty$, $\frac{X_1+\cdots+X_n}{n^{1/2}}$ converges in distribution to a constant random variable (with variance zero), unless $\alpha = 2$. In the remaining case $\alpha = 2$, we know from the Lévy Continuity Theorem that, if $\phi_{X_1}(t) = e^{-t^2}$, then X_1 is a mean zero Gaussian random variable, and likewise the same holds for $\frac{X_1+\cdots+X_n}{n^{1/2}}$. so, $\frac{X_1+\cdots+X_n}{n^{1/2}}$ is a mean zero Gaussian (with the same nonzero variance), for all $n \geq 1$.