Probability, 407, Fall 2023, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may not use your books or notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 120 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Do not write in the table to the right. Good luck![a]

1. (10 points) Let X be a random variable uniformly distributed in $[0, 1]$.

   (That is, X has PDF $f_X(x) = 1$ when $x \in [0, 1]$, and $f_X(x) = 0$ when $x \notin [0, 1]$.)

   Let Y be a random variable uniformly distributed in $[0, 1]$.

   Assume that X and Y are independent.

   - Compute $\mathbf{P}(X > 3/4)$.
   - Compute $\mathbf{E}X$.
   - Compute $\mathbf{P}(X + Y \leq 1/2)$.

   In all cases, simplify your answer to the best of your ability.

   [This was modified from a practice exam.]

2. (10 points) Let X and Y be independent random variables. Assume that X is uniformly distributed in $[-1, 1]$, and Y is uniformly distributed in $[-1, 1]$.

   - Compute $\mathbf{E}(X^2Y)$.
   - Compute $\mathbf{P}(X^2 + Y^2 \geq 1)$.

   In all cases, simplify your answer to the best of your ability.

   Simplify your answer to the best of your ability.

   [This was modified from a practice exam.]

3. (10 points) Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs twenty times, so that twenty spheres have been placed in bins. (All of the sphere placements up to this point are independent of each other).

   Suppose you now flip a fair coin. (A fair coin has probability $1/2$ of landing heads, and probability $1/2$ of landing tails). (The coin flip result is independent of all of the sphere placements.) If the coin lands heads, you then place another ten spheres randomly into the bins (with each sphere being equally likely to appear in any of the five bins).

   What is the expected number of empty bins?

   Simplify your answer to the best of your ability. (As usual, show your work.)

   [This was repeated from a previous exam.]

4. (10 points) Let X, Y be independent standard Gaussian random variables. (That is, X has PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\forall\, x \in \mathbf{R}$.)

   Let $Z := X/\left|Y\right|$.

   Find the PDF of Z.

   (Justify your answer.) (Simplify your answer to the best of your ability.) (PDF is an acronym for: Probability Density Function.)

   [This example was in the notes]

5. (10 points) Let $X_1, X_2, \ldots$ be independent random variables, each with exponential distribution with parameter $\lambda = 1$. (That is, X_1 has PDF $f_X(x) = e^{-x}$ when $x \geq 0$, and $f_X(x) = 0$ for $x < 0$.)

   For any $n \geq 1$, let $Y_n := \max(X_1, \ldots, X_n)$. Let $0 < a < 1 < b$.

   - Show that $\lim_{n\to\infty} \mathbf{P}(Y_n \leq a \log n) = 0$ and $\lim_{n\to\infty} \mathbf{P}(Y_n \leq b \log n) = 1$.
   - Show that, for all $\varepsilon > 0$,

   $$\lim_{n\to\infty} \mathbf{P}\left(\left|\frac{Y_n}{\log n} - 1\right| > \varepsilon\right) = 0.$$

   [This was a repeated homework question.]

6. (10 points) Suppose you flip a fair coin 120 times. During each coin flip, this coin has probability $1/2$ of landing heads, and probability $1/2$ of landing tails.

   Let A be the event that you get more than 90 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{60}.$$

   [This appeared on practice exams and was related to some homework problems.]

7. (10 points) Let $X_1, X_2, \ldots$ be i.i.d. (independent identically distributed) random variables. Assume that $\mathbf{E}\,|X_1| < \infty$ and $\mathrm{Var}(X_1) = 0$. Denote $\mu := \mathbf{E}X_1$.

   Does the random variable
   $$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}}$$
   converge in distribution to some random variable Z as $n \to \infty$?

   If so, what is the CDF of Z?

   (Here CDF denotes cumulative distribution function.)

   (Justify your answer.)

   [This is the zero variance case of the CLT]

8. (10 points) Let $X_1, X_2, \ldots$ be i.i.d. (independent identically distributed) random variables. Fix a real number $0 < \alpha \leq 2$. Assume X_1 has a characteristic function given by

$$\phi_{X_1}(t) = \mathbf{E}e^{\sqrt{-1}\,tX_1} = e^{-|t|^\alpha}, \qquad \forall\, t \in \mathbf{R}$$

- Prove that

  $$\phi_{\left(\frac{X_1+\cdots+X_n}{n^{1/\alpha}}\right)}(t) = \phi_{X_1}(t), \qquad \forall\, t \in \mathbf{R}.$$

  (Here $n \geq 1$ is a positive integer.)

- For what values of $0 < \alpha \leq 2$ does the random variable

  $$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

  converge in distribution to a Gaussian random variable with positive variance, as $n \to \infty$? (Justify your answer.)

(Scratch paper)

(Scratch paper)