Probability, 407, Fall 2020, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 120 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Do not write in the table to the right. Good luck![a]

1. (10 points) Let X and Y be independent random variables. Assume that X is uniformly distributed in $[0, 2]$, and Y is uniformly distributed in $[0, 4]$.

   Find the density of $X + Y$.

   Simplify your answer to the best of your ability.

2. (10 points) Let X, Y and Z be independent random variables. Suppose X and Y are exponential random variables with parameter 1. That is,

$$f_X(x) = \begin{cases} 0 & , \text{if } x < 0 \\ e^{-x} & , \text{if } x \geq 0. \end{cases} \qquad f_Y(y) = f_X(y), \ \forall\, y \in \mathbf{R}.$$

Suppose Z is uniformly distributed in $[0, 1]$.

Let $W = [\max(X, Y, Z^2)]^3$. Find the density of W.

Simplify this expression to the best of your ability.

3. (10 points) Give an example of a random variable X such that the following conditions all hold simultaneously for X:

   - X is **NOT** a discrete random variable.
   - X is **NOT** a continuous random variable.

   Prove that X satisfies these properties.

4. (10 points) Suppose you take a fair three-sided die and you roll it 90 times. For each roll of the die, it has $1/3$ probability of rolling a $1, 2$ or 3.

   Let A be the event that the sum of the die rolls is greater than 240. Show that

$$\mathbf{P}(A) \leq \frac{1}{120}.$$

5. (10 points) Let X, Y and Z be independent random variables that are each uniformly distributed in the interval $[0, 3]$. (That is, X is uniformly distributed in $[0, 3]$, Y is uniformly distributed in $[0, 3]$, and Z is uniformly distributed in $[0, 3]$.) Compute

$$\mathbf{P}(X + Y + Z \leq 1).$$

Your final answer should be a ratio of integers.

6. (10 points) A red cube, a green cube, and a blue cube are each put in a bowl. One cube is removed from the bowl, uniformly at random, and that cube is set on the table. Then, another cube is removed from the bowl, uniformly at random, and that cube is set on the table. In this way, two cubes have been randomly removed from the bowl.

   Let R be a random variable such that $R = 1$ if the red cube is removed from the bowl and $R = 0$ otherwise.

   Let G be a random variable such that $G = 1$ if the green cube is removed from the bowl and $G = 0$ otherwise.

   - Compute $\text{cov}(R, G)$.
   - Compute $\text{var}(R)$.
   - Compute $\text{var}(3R + 6G)$.

   In all cases, simplify your answer to the best of your ability.

7. (10 points) Let $0 < p < 1$ with $p \neq 1/2$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables such that $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1 - p$.

   Let $S_0 = 0$ and for any $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Show that

   $$\mathbf{P}(S_{2n} = 0 \text{ for infinitely many } n \geq 1) = 0.$$

   (Hint: compute $\mathbf{P}(S_{2n} = 0)$ exactly. Then estimate $\sum_{n=0}^{\infty} \mathbf{P}(S_{2n} = 0)$. You can freely use Stirling's formula, which says that $n! \sim \sqrt{2\pi n}(n/e)^n$, i.e. $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1$.)

8. (10 points) In this Exercise we show a few steps towards proving the De Moivre-Laplace Theorem. Recall that $M_X(t) = \mathbf{E}e^{tX}$ is the moment generating function of a random variable X for any $t \in \mathbf{R}$. Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$, so $\mathbf{E}X_1 = 0$ and $\mathrm{var}(X_1) = 1$.

   - Find an explicit formula for $M_{X_1}(t)$, for any $t \in \mathbf{R}$. Simplify your answer to the best of your ability.
   - Using a Taylor expansion near $t = 0$, show that $|M_{X_1}(t) - [1 + t^2/2]| \leq 100t^3$ for all t near 0. (Hint: Taylor's Theorem with error term says that

$$f(t) = f(0) + tf'(0) + (t^2/2)f''(0) + (1/2)\int_0^t (t-s)^2 f'''(s)ds,$$

     for all t near 0. You can freely use Taylor's Theorem.)
   - Find an explicit formula for $M_{(X_1+\cdots+X_n)/\sqrt{n}}(t)$, for any $t \in \mathbf{R}$. Simplify your answer to the best of your ability.
   - For any $t \in \mathbf{R}$, compute

$$\lim_{n\to\infty} M_{(X_1+\cdots+X_n)/\sqrt{n}}(t).$$

     Relate the answer you get to the moment generating function of a standard Gaussian random variable.

(Scratch paper)

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