Probability, 407, Fall 2020, USC — Instructor: Steven Heilman

Name: ____________ USC ID: ____________ Date: ____________

Signature: ____________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 120 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Let X and Y be independent random variables.

   Suppose X is uniformly distributed in the integers $\{1, 2, 3, \ldots, 100\}$.

   Suppose Y is uniformly distributed in the integers $\{-100, -99, -98, \ldots, -1\}$.

   What is the PMF of $X + Y$? Simplify your answer to the best of your ability.

2. (10 points) Give an example of the joint density of two continuous random variables X and Y such that: X and Y are **NOT** independent.

   Prove that the X and Y you find are not independent.

3. (10 points) Let X and Y be independent random variables. Suppose X is uniformly distributed in $[0, 1]$. Suppose Y is an exponential random variable with parameter 1. That is, Y has density

$$f_Y(y) = \begin{cases} 0 & \text{, if } y < 0 \\ e^{-y} & \text{, if } y \geq 0. \end{cases}$$

Let $Z = \max\big(X(1-X)\,,\,Y\big)$ be the maximum of $X(1-X)$ and Y.

Find f_Z, the density function of Z.

Simplify your answer to the best of your ability.

4. (10 points)

- Find a random variable X such that

$$\mathbf{P}(|X| \geq 3) = \frac{\mathbf{E}\,|X|}{3}.$$

  Prove that X satisfies this property. (Hint: can X take only one value?)

- Find a random variable Y such that

$$\mathbf{P}(|Y - \mathbf{E}Y| \geq 2) = \frac{\mathrm{var}(Y)}{4}.$$

  Prove that Y satisfies this property.

5. (10 points) Let X and Y be random variables. Let t be a constant. Suppose these random variables have joint density function

$$f_{X,Y}(x,y) = \begin{cases} tx^2y^2 & \text{, if } -1 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{, otherwise.} \end{cases}$$

- Find the constant t such that $f_{X,Y}$ is a joint probability density function.
- Let f_Y be the marginal density of Y. Show that its third derivative satisfies

$$\frac{d^3}{dy^3} f_Y(y) = 0, \qquad \forall\, y \in (0,1).$$

- Write a formula that computes $\mathbf{P}(X > Y)$ using integrals. You do **NOT** have to simplify this formula. Your final answer **must** be an integral of the following form:

$$\int_{x=(\cdots)}^{x=(\cdots)} \int_{y=(\cdots)}^{y=(\cdots)} (\text{some function})\, dy dx.$$

6. (10 points) Let X and Y be independent random variables. Suppose X has characteristic function (Fourier Transform)

$$\phi_X(t) = e^{-t^2}, \qquad \forall\, t \in \mathbf{R}.$$

(Recall that $\phi_X(t) = \mathbf{E}e^{itX}$ where $i = \sqrt{-1}$, for any $t \in \mathbf{R}$.) Suppose Y has moment generating function

$$M_Y(t) = 1 + t^4, \qquad \forall\, t \in \mathbf{R}.$$

(Recall that $M_Y(t) = \mathbf{E}e^{tY}$ for any $t \in \mathbf{R}$.)

Compute $\mathbf{E}\Big[(X+Y)^2\Big]$.

7. (10 points) Consider a population of 30,000 people, where half of them are given a vaccine for a disease. Suppose all 30,000 people are exposed to a virus causing the disease. We observe that 90 of the unvaccinated people catch the disease, while 5 of the vaccinated people catch the disease.

   Consider the following statement:

   "The number of infections of vaccinated people, divided by the number of infections of unvaccinated people, is less than 15/100."

   Is this statement true with greater than 90% certainty? Justify your answer.

   (Assume that each person's ability to catch the disease is independent of each other person's ability to catch the disease.)

   (Hint: the estimated probability of a vaccinated person getting the disease is $5/15,000$, and the estimated probability of an unvaccinated person getting the disease is $90/15,000$.)

   (Hint: use the Central Limit Theorem. If Z is a standard Gaussian, then $\mathbf{P}(|Z| \leq 2) \approx .9545$. Also, $\sqrt{5} \approx 2.23$, $\sqrt{90} \approx 9.5$.)

8. (10 points) Suppose you are flipping a fair coin, so that each flip of the coin has probability $1/2$ of landing heads, and probability $1/2$ of landing tails. What is the expected number of coin flips that you have to make until you see two consecutive heads appear? (That is, you keep flipping the coin until you see two heads in a row, at which point you stop flipping the coin any more, and you count the total number of coin flips you have made.) (Hint: condition on the first two coin flips.)

   (Simplify your final answer to the best of your ability.)

(Scratch paper)

(Extra Scratch paper)