Probability, 407, Fall 2020, USC   Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) Let X be a continuous random variable with probability density function f_X. Then $f_X(x) \le 1$ for all $x \in \mathbf{R}$.

   TRUE FALSE (circle one)

   (b) (2 points) When X is a continuous random variable, there is a continuous function $f_X \colon \mathbf{R} \to [0, \infty)$ such that, for any $-\infty \le a \le b \le \infty$,

   $$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

   TRUE FALSE (circle one)

   (c) (2 points) Let X be a continuous random variable with PDF $f_X \colon \mathbf{R} \to [0, \infty)$. Then, for any $t \in \mathbf{R}$,

   $$\frac{d}{dt}\mathbf{P}(X \le t) = f_X(t).$$

   TRUE FALSE (circle one)

(d) (2 points) Let X and Y be discrete random variables. Then

$$\mathbf{E}(XY) = (\mathbf{E}X)(\mathbf{E}Y).$$

TRUE FALSE (circle one)

(e) (2 points) Let $A_1, \ldots, A_n$ be disjoint events in a sample space Ω. That is, $A_i \cap A_j = \emptyset$ whenever $i, j \in \{1, \ldots, n\}$ satisfy $i \neq j$. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(A_i) > 0$ for all $1 \leq i \leq n$. Let $X \colon \Omega \to \mathbf{R}$ be a discrete random variable. Then

$$\mathbf{E}X = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{E}(X|A_i).$$

TRUE FALSE (circle one)

2. (10 points) Let X be a discrete random variable such that

$$\mathbf{P}(X = 1) = \mathbf{P}(X = 2) = \mathbf{P}(X = 3) = 1/6, \qquad \text{and}$$

$$\mathbf{P}(X = -1) = \mathbf{P}(X = -2) = \mathbf{P}(X = -3) = 1/6.$$

Compute the following quantities:

- $\mathbf{E}X$
- $\mathbf{E}(X^2)$
- $\mathrm{var}(X)$

Simplify your answers to the best of your ability. (As usual, show your work.)

3. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$, so that X has PDF

$$f_X(x) = e^{-x}, \qquad \forall\, x \geq 0,$$

and $f_X(x) = 0$ for all $x < 0$.

Compute the following quantities:

- $\mathbf{E}X$
- $\mathbf{P}(X > 1)$

Simplify your answers to the best of your ability. (As usual, show your work.)

4. (10 points) Let X and Y be discrete random variables such that $|X| \leq 10$ and $|Y| \leq 10$. Recall that $\text{cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}X)(Y - \mathbf{E}Y)]$.

   Prove or disprove the statement below. (In the case that you disprove the statement, it suffices to find a counterexample and explain your reasoning.)

   Statement: If $\text{cov}(X, Y) = 0$, then X and Y are independent.

5. (10 points) Let X be a standard Gaussian random variable, so that X has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \forall\, x \in \mathbf{R}.$$

Let Z be the random variable defined by

$$Z = X^4.$$

What is the PDF of Z? (As usual, justify your answer.)

(Scratch paper)