407 Midterm 2 Solutions[1]

1. Question 1

Let X be a binomial random variable with parameters $n = 2$ and $p = 1/4$ so that for all integers k satisfying $0 \leq k \leq 2$,

$$\mathbf{P}(X = k) = \binom{2}{k} p^k (1-p)^{n-k} = \frac{2!}{k!(2-k)!}(1/4)^k (3/4)^{2-k}.$$

Compute the following quantities: $\mathbf{E}X$, $\mathbf{E}(X^2)$, $\mathrm{var}(X)$

Simplify your answers to the best of your ability. (As usual, show your work.)

Solution. We have $\mathbf{P}(X = 0) = (3/4)^2 = 9/16$, $\mathbf{P}(X = 1) = 2(1/4)(3/4) = 6/16$, $\mathbf{P}(X = 2) = (1/4)^2 = 1/16$, so

$$\mathbf{E}X = \sum_{x \in \mathbf{R}} x p_X(x) = 0(9/16) + 1(6/16) + 2(1/16) = 8/16 = 1/2.$$

$$\mathbf{E}X^2 = \sum_{x \in \mathbf{R}} x^2 p_X(x) = 0^2(9/16) + 1^2(6/16) + 2^2(1/16) = 10/16.$$

$$\mathrm{var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 10/16 - (1/2)^2 = (10-4)/16 = 6/16 = 3/8.$$

2. Question 2

Let X be a standard Gaussian random variable, so that X has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \forall\, x \in \mathbf{R}.$$

Compute the following quantities: $\mathbf{P}(X = 10)$, $\mathbf{P}(X > 0)$.

Simplify your answers to the best of your ability. (As usual, show your work.)

Solution. By definition of PDF,

$$\mathbf{P}(X = 10) = \int_{10}^{10} f_X(x)dx = 0.$$

Also, using symmetry of the Gaussian PDF,

$$\mathbf{P}(X > 0) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2} \cdot 1 = 1/2.$$

3. Question 3

Let $0 < p < 1$. Let X be a geometric random variable with parameter p, so that, for any positive integer k,

$$\mathbf{P}(X = k) = (1-p)^{k-1} p.$$

In class, we computed $\mathbf{E}X = 1/p$ and $\mathbf{E}X^2$ using a conditioning argument. For example, we showed $\mathbf{E}(X^2 | X = 1) = 1$ and $\mathbf{E}(X^2 | \, X > 1) = 1 + 2\mathbf{E}X + \mathbf{E}X^2$. We then solved for $\mathbf{E}X^2$ to get $\mathbf{E}X^2 = (2/p^2) - 1/p$.

Using this same conditioning argument (i.e. by conditioning on $X = 1$ and on $X > 1$), compute $\mathbf{E}(X^3)$. *Solution.* From the Total Expectation Theorem,

$$\mathbf{E}X^3 = \mathbf{E}(X^3|X=1)\mathbf{P}(X=1) + \mathbf{E}(X^3|X>1)\mathbf{P}(X>1) = \mathbf{E}(X^3|X=1)p + \mathbf{E}(X^3|X>1)(1-p).$$

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When $X = 1$, $X^3 = 1$ also, so $\mathbf{E}(X^3|X = 1) = 1$. Also, given that $X > 1$, i.e. $X \neq 1$, this is equivalent to knowing that the first coin flip (defining X) is tails, and then starting over again our wait for the first heads coin flip. That is, $\mathbf{E}(X^3|X > 1) = \mathbf{E}((X + 1)^3) = \mathbf{E}X^3 + 3\mathbf{E}X^2 + 3\mathbf{E}X + 1$. In summary,

$$\mathbf{E}X^3 = p + (1 - p)[\mathbf{E}X^3 + 3\mathbf{E}X^2 + 3\mathbf{E}X + 1].$$

That is,

$$\mathbf{E}X^3[1 - (1 - p)] = p + (1 - p)[3\mathbf{E}X^2 + 3\mathbf{E}X + 1]$$

Substituting in our known values of the first and second moment,

$$\mathbf{E}X^3 = 1 + \frac{1}{p}(1 - p)[3((2/p^2) - 1/p) + 3(1/p) + 1].$$

4. Question 4

Let X be binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 4$ and $\mathbf{E}Y^2 = 32$.

Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

Let W be a discrete random variable such that $\mathbf{P}(W = -1) = 1/2$ and $\mathbf{P}(W = 1) = 1/2$, so that $\mathbf{E}W = 0$ and $\mathbf{E}W^2 = 1$.

Assume that X, Y and Z are all independent. Compute

$$\mathbf{E}(1 + W^{100} + W^{50}XYZ^2).$$

(You **cannot** assume that W is independent of X, Y, Z.)

Solution. By Definition of W, $W^{50} = W^{100} = 1$, so

$$\mathbf{E}(1 + W^{100} + W^{50}XYZ^2) = \mathbf{E}(1 + 1 + XYZ^2).$$

Since X, Y, Z are independent, we have $\mathbf{E}(XYZ^2) = \mathbf{E}X\mathbf{E}Y\mathbf{E}Z^2 = 1\cdot 4\cdot 2 = 8$. Finally, since the expected value of a sum is the sum of expected values, the final answer is $1+1+8 = 10$.

5. Question 5

Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let $Y = \min(X_1, \ldots, X_n)$ be the minimum of $X_1, \ldots, X_n$. What is the PDF of Y? *Solution.* As shown in class, the event $Y \geq t$ is equal to the event $X_1 \geq t, \ldots, X_n \geq t$. So, $\mathbf{P}(Y \geq t) = [\mathbf{P}(X_1 \geq t)]^n = [\int_t^\infty e^{-x^2/2}dx/\sqrt{2\pi}]^n$ for any $t \in \mathbf{R}$. We can then get the density of Y, since

$$f_Y(t) = \frac{d}{dt}\mathbf{P}(Y \leq t) = \frac{d}{dt}[1 - \mathbf{P}(Y \geq t)] = -\frac{d}{dt}[\int_t^\infty e^{-x^2/2}dx/\sqrt{2\pi}]^n$$
$$= n\left(\int_t^\infty e^{-x^2/2}dx/\sqrt{2\pi}\right)^{n-1} e^{-t^2/2}/\sqrt{2\pi}.$$