Probability, 407, Fall 2020, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Problem	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
Total:	48	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) The negation of the statement
   "For every integer j, we have $j^2 + 1 > 0$" is:
   "There exists and integer j such that $j^2 + 1 \leq 0$."

   TRUE FALSE (circle one)

   (b) (2 points) Let $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. For any $A \subseteq \Omega$, define $\mathbf{P}(A)$ to be the number of elements in A. Then $\mathbf{P}$ is a probability law on Ω.

   TRUE FALSE (circle one)

   (c) (2 points) Let A, B be subsets of a sample space Ω. Then

   $$A = (A \cap B) \cup (A \cap B^c).$$

   TRUE FALSE (circle one)

   (d) (2 points) Let $A_1, \ldots, A_n$ be disjoint events in a sample space Ω. That is, $A_i \cap A_j = \emptyset$ whenever $i, j \in \{1, \ldots, n\}$ satisfy $i \neq j$. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(A_i) > 0$ for all $1 \leq i \leq n$. Let $B \subseteq \Omega$. Then

   $$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(B|A_i).$$

   TRUE FALSE (circle one)

2. (10 points) Prove the following equality for subsets of the real line:

$$\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\ \right] = (0, 1).$$

3. (10 points) A single fair 100-sided die has each of its faces labeled with exactly one integer between and including 1 and 100. Each face is equally like to be rolled.

   Suppose you have three fair 100-sided dice. After rolling these three dice, what is the probability that the sum of the rolls of the three dice is 52?

4. (10 points) Let A, B, C, D be events in a sample space Ω. Prove:

$$\mathbf{P}(A \cap B \cap C \cap D) = \mathbf{P}(A) \cdot \mathbf{P}(B|A) \cdot \mathbf{P}(C|\, A \cap B) \cdot \mathbf{P}(D|\, A \cap B \cap C).$$

5. (10 points) Let A, B, C be subsets of a sample space Ω. Suppose we know that

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

Are the sets A, B, C independent?

If the sets A, B, C are independent, prove it.

If it is possible that the sets A, B, C are not be independent, provide a counterexample and explain your reasoning. (Make sure to specify $\Omega, \mathbf{P}, A, B$ and C.)

(Scratch paper)