407 Midterm 1 Solutions[1]

1. Question 1

TRUE/FALSE

(a) The negation of the statement "There exists an integer j such that $j^3 - j < 7$" is: "For every integer j, we have $j^3 - j \geq 7$."

TRUE, by the rules of negation, "There exists" is negated to "For every," and the inequality $<$ is negated to $\geq$.

(b) Let $\mathbf{P}$ be the uniform probability law on $[0,1]$. Let $x_1, x_2, \ldots \in [0,1]$ be a countable set of distinct points. Then

$$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = 0.$$

TRUE. By the definition of $\mathbf{P}$, $\mathbf{P}(\{x_n\}) = 0$ for all $n \geq 1$. So, from Axiom (ii) for probability laws,

$$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = \sum_{n=1}^{\infty}\mathbf{P}(\{x_n\}) = \sum_{n=1}^{\infty} 0 = 0.$$

(c) Let A be a subset of a sample space Ω. Then

$$A \cap A^c = \emptyset.$$

TRUE. Let $x \in A^c$. Then $x \notin A$, by definition of A^c. So, by definition of intersection, $A \cap A^c = \emptyset$.

(d) Let A, B, C be subsets of a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(C) > 0$. Then

$$\mathbf{P}(A \cup B \,|\, C) = \mathbf{P}(A \,|\, C) + \mathbf{P}(B \,|\, C) - \mathbf{P}(A \cap B \,|\, C).$$

TRUE. From Proposition 2.38 in the notes, $\mathbf{P}(\cdot|C)$ is a probability law on Ω. So, this statement follows from Proposition 2.33, part (ii) in the notes.

2. Question 2

Prove the following assertion by induction on n:

For any positive integer n, we have $1 + 2 + 3 + \cdots + n = n(n+1)/2$.

Solution. We first check the base case $n = 1$. In this case, we have $n(n+1)/2 = 1(2)/2 = 1$. So, the base case holds. We now check the inductive step. Suppose $1 + 2 + 3 + \cdots + n = n(n+1)/2$ for some positive integer n. We now consider the case $n+1$. Then

$$\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= [1 + 2 + \cdots + n] + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{, by the inductive hypothesis} \\
&= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}$$

That is, the desired assertion holds in the case $n+1$. Since the inductive step has concluded, the assertion holds for all positive integers n.

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3. Question 3

An urn contains three red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional red cube is put into the urn, and then a second cube is removed from the urn.

- What is the probability that the second cube removed from the urn is red?
- If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

Solution. Let A be the event that the first cube removed is red, and let B be the event that the first cube removed is blue. Let C be the event that the second cube removed from the urn is red. Then $A \cap B = \emptyset$ and $A \cup B = \Omega$, so the Total Probability Theorem says

$$\mathbf{P}(C) = \mathbf{P}(C|A)\mathbf{P}(A) + \mathbf{P}(C|B)\mathbf{P}(B) = (1/2)(3/5) + (4/6)(2/5) = 3/10 + 8/30 = 17/30.$$

Now, using that $\mathbf{P}(C) = 17/30$, we have by Bayes' rule

$$\mathbf{P}(B|C) = \mathbf{P}(C|B)[\mathbf{P}(B)/\mathbf{P}(C)] = (4/6)(2/5)(30/17) = 8/17.$$

4. Question 4

Let $\Omega = \mathbf{R}^2$, and define

$$\mathbf{P}(A) = \frac{1}{2\pi} \iint_A e^{-\frac{(x^2+y^2)}{2}}\, dxdy$$

Suppose $A = \{(x,y) \in \Omega \colon -1 \leq x \leq 1\}$ and $B = \{(x,y) \in \Omega \colon x^2 + y^2 \leq 1\}$.
Are A and B independent? Prove your assertion.

Solution. A and B are not independent. Since $B \subseteq A$ and $A \setminus B$ is nonempty, we have $A \cap B = B$, so that

$$\mathbf{P}(A \cap B) = \mathbf{P}(B)$$

Meanwhile, $\mathbf{P}(A) < 1$ since

$$\mathbf{P}(A) = \frac{1}{2\pi} \iint_A e^{-\frac{(x^2+y^2)}{2}}\, dxdy = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1,$$

Moreover, $\mathbf{P}(B) > 0$ since

$$\mathbf{P}(B) = \frac{1}{2\pi} \iint_{x^2+y^2 \leq 1} e^{-\frac{(x^2+y^2)}{2}}\, dxdy \geq \frac{1}{2\pi} \iint_{x^2+y^2 \leq 1} e^{-1/2}\, dxdy = \frac{1}{2\pi} e^{-1/2} \pi > 0.$$

Combining $\mathbf{P}(B) > 0$ with $0 \leq \mathbf{P}(A) < 1$, we have

$$\mathbf{P}(A)\mathbf{P}(B) < \mathbf{P}(B) = \mathbf{P}(A \cap B).$$

That is, $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$. That is, A and B are not independent.

5. Question 5

Suppose you have \$100, and you need to come up with \$400. You are a terrible gambler but you decide you need to gamble your money to get \$200. For any amount of money M, if you bet $\$M$, then you win $\$M$ with probability .3, and you lose $\$M$ with probability .7. (If you run out of money, you stop gambling, and if you ever have at least \$400, then you stop gambling.) Consider the following two possible strategies for gambling:

Strategy 1. Bet the amount of money that you have, each time.

Strategy 2. Make a small bet of \$10 each time.

Explain which strategy is better. That is, explain which strategy has a higher probability of getting \$400.

Solution. Strategy 1 is much better. The probability of reaching \$400 with consecutive wins is $(.3)^2$, since if you win each time, your sequence of monetary holdings would be: \$100, \$200, \$400. So, with probability at least $(.3)^2$, you will reach \$400 in winnings. On the other hand, your ability to make it to \$400 with Strategy 2 is astronomically low. The Gambler's Ruin problem from Example 2.53 in the notes shows that the probability of reaching \$400 with \$10 bets is the same as: starting with \$10, making \$1 bets, and stopping when you reach \$0 or \$40. The probability of reaching \$40 is

$$\frac{\left(\frac{.7}{.3}\right)^{10}-1}{\left(\frac{.7}{.3}\right)^{40}-1} \leq \frac{3^{10}}{2^{40}} = \frac{3^{10}}{(2^2)^{10}2^{20}} \leq 2^{-20}.$$

And 2^{-20} is much less than $(.3)^4$. That is, Strategy 1 is far superior to Strategy 2.