Probability, 407, Fall 2020, USC — Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Problem	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
Total:	48	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) The negation of the statement
   "There exists an integer j such that $j^3 - j < 7$" is:
   "For every integer j, we have $j^3 - j \geq 7$."

   TRUE FALSE (circle one)

   (b) (2 points) Let $\mathbf{P}$ be the uniform probability law on $[0,1]$. Let $x_1, x_2, \ldots \in [0,1]$ be a countable set of distinct points. Then

   $$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = 0.$$

   TRUE FALSE (circle one)

   (c) (2 points) Let A be a subset of a sample space Ω. Then

   $$A \cap A^c = \emptyset.$$

   TRUE FALSE (circle one)

   (d) (2 points) Let A, B, C be subsets of a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(C) > 0$. Then

   $$\mathbf{P}(A \cup B \,|\, C) = \mathbf{P}(A \,|\, C) + \mathbf{P}(B \,|\, C) - \mathbf{P}(A \cap B \,|\, C).$$

   TRUE FALSE (circle one)

2. (10 points) Prove the following assertion by induction on n:

   For any positive integer n, we have $1 + 2 + 3 + \cdots + n = n(n+1)/2$.

3. (10 points) An urn contains three red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional red cube is put into the urn, and then a second cube is removed from the urn.

   - What is the probability that the second cube removed from the urn is red?
   - If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

4. (10 points) Let $\Omega = \mathbf{R}^2$, and define

$$\mathbf{P}(A) = \frac{1}{2\pi} \iint_A e^{-\frac{(x^2+y^2)}{2}} \, dx dy$$

Suppose $A = \{(x,y) \in \Omega \colon -1 \leq x \leq 1\}$ and $B = \{(x,y) \in \Omega \colon x^2 + y^2 \leq 1\}$. Are A and B independent? Prove your assertion.

5. (10 points) Suppose you have \$100, and you need to come up with \$400. You are a terrible gambler but you decide you need to gamble your money to get \$200. For any amount of money M, if you bet \$$M$, then you win \$$M$ with probability .3, and you lose \$$M$ with probability .7. (If you run out of money, you stop gambling, and if you ever have at least \$400, then you stop gambling.) Consider the following two possible strategies for gambling:

   Strategy 1. Bet the amount of money that you have, each time.

   Strategy 2. Make a small bet of \$10 each time.

   Explain which strategy is better. That is, explain which strategy has a higher probability of getting \$400.

(Scratch paper)