Please provide complete and well-written solutions to the following exercises.

Due February 14, in the discussion section.

Homework 5

Exercise 1. Give an explicit example of a Markov chain where every state has period 100.

Exercise 2. Let Ω be a finite state space. This exercise demonstrates that the total variation distance is a metric. That is, the following three properties are satisfied:

- $||\mu - \nu||_{\mathrm{TV}} \geq 0$ for all probability distributions μ, ν on Ω, and $||\mu - \nu||_{\mathrm{TV}} = 0$ if and only if $\mu = \nu$.
- $||\mu - \nu||_{\mathrm{TV}} = ||\nu - \mu||_{\mathrm{TV}}$
- $||\mu - \nu||_{\mathrm{TV}} \leq ||\mu - \eta||_{\mathrm{TV}} + ||\eta - \nu||_{\mathrm{TV}}$ for all probability distributions μ, ν, η on Ω.

(Hint: you may want to use the triangle inequality for real numbers: $|x - y| \leq |x - z| + |z - y|$, $\forall\, x, y, z \in \mathbf{R}$.)

Exercise 3. Let μ, ν be probability distributions on a finite state space Ω. Then

$$||\mu - \nu||_{\mathrm{TV}} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|\,.$$

(Hint: consider the set $A = \{x \in \Omega \colon \mu(x) \geq \nu(x)\}$.)

Exercise 4. Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbb{Z}$. Show that $\mathbf{P}_0(X_n = 0)$ decays like $1/\sqrt{n}$ as $n \to \infty$. That is, show

$$\lim_{n \to \infty} \sqrt{2n}\, \mathbf{P}_0(X_{2n} = 0) = \sqrt{\frac{2}{\pi}}.$$

Also, show the upper bound

$$\mathbf{P}_0(X_n = k) \leq \frac{10}{\sqrt{n}}, \qquad \forall\, n \geq 0,\, k \in \mathbb{Z}.$$

(Hint 1: first consider the case $n = 2r$ for $r \in \mathbb{Z}$. It may be helpful to show that $\binom{2r}{r+j}$ is maximized when $j = 0$. To eventually deal with k odd, just condition on the first step of the walk.)

(Hint 2: you can freely use **Stirling's formula**:

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1.$$

Or, there is a more precise estimate: for any $n \geq 3$, there exists $1/(12n+1) \leq \varepsilon_n \leq 1/(12n)$ such that

$$n! = \sqrt{2\pi} e^{-n} n^{n+1/2} e^{\varepsilon_n}.)$$

Exercise 5. Show that every state in the simple random walk on $\mathbb{Z}$ is recurrent. (You should show this statement for any starting location of the Markov chain.)

Then, find a nearest-neighbor random walk on $\mathbb{Z}$ such that every state is transient.

Exercise 6. For the simple random walk on $\mathbb{Z}$, show that $\mathbf{E}_0 T_0 = \infty$. Conclude that, for any $x, y \in \mathbb{Z}$, $\mathbf{E}_x T_y = \infty$.

Exercise 7. Let $(X_0, X_1, \ldots)$ be the "corner walk" on $\mathbb{Z}^2$. The transitions are described as follows. From any point $(x, y) \in \mathbb{Z}^2$, the Markov chain adds any of the following four vector to (x, y) each with probability $1/4$: $\{(1, 1,), (1, -1), (-1, 1), (-1, -1)\}$. Using that the coordinates of this walk are each independent simple random walks on $\mathbb{Z}$, conclude that there exists $c > 0$ such that

$$\lim_{n\to\infty} n\mathbf{P}_{(0,0)}(X_{2n} = (0, 0)) = c.$$

That is, $\mathbf{P}_{(0,0)}(X_{2n} = (0, 0))$ is about c/n, when n is large.

Now, note that the usual nearest-neighbor simple random walk on $\mathbb{Z}^2$ is a rotation of the corner walk by an angle of $\pi/4$. So, the above limiting statement also holds for the simple random walk on $\mathbb{Z}^2$.