Please provide complete and well-written solutions to the following exercises.

Due January 17, in the discussion section.

Homework 1

Exercise 1. Let $\phi\colon \mathbf{R}\to\mathbf{R}$. Show that ϕ is convex if and only if: for any $y\in\mathbf{R}$, there exists a constant a and there exists a function $L\colon\mathbf{R}\to\mathbf{R}$ defined by $L(x)=a(x-y)+\phi(y)$, $x\in\mathbf{R}$, such that $L(y)=\phi(y)$ and such that $L(x)\leq\phi(x)$ for all $x\in\mathbf{R}$. (In the case that ϕ is differentiable, the latter condition says that ϕ lies above all of its tangent lines.)

(Hint: Suppose ϕ is convex. If x is fixed and y varies, show that $\frac{\phi(y)-\phi(x)}{y-x}$ increases as y increases. Draw a picture. What slope a should L have at x?)

Exercise 2. Let X,Y be positive random variables on a sample space $\mathcal{C}$. Assume that $X(c)\geq Y(c)$ for all $c\in\mathcal{C}$. Prove that $\mathbf{E}X\geq\mathbf{E}Y$.

More generally, if $X\leq Y$, $\mathbf{E}\,|X|<\infty$ and $\mathbf{E}\,|Y|<\infty$, show that $\mathbf{E}X\leq\mathbf{E}Y$.

Exercise 3. Let $X_1,\ldots,X_n$ be discrete random variables. Assume that

$$\mathbf{P}(X_1=x_1,\ldots,X_n=x_n)=\prod_{i=1}^{n}\mathbf{P}(X_i=x_i),\qquad\forall\,x_1,\ldots,x_n\in\mathbf{R}.$$

Show that $X_1,\ldots,X_n$ are independent.

Exercise 4. Let $X_1,\ldots,X_n$ be continuous random variables with joint PDF $f\colon\mathbf{R}^n\to[0,\infty)$. Assume that

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)=\prod_{i=1}^{n}f_{X_i}(x_i),\qquad\forall\,x_1,\ldots,x_n\in\mathbf{R}.$$

Show that $X_1,\ldots,X_n$ are independent.

Exercise 5. Estimate the probability that 1000000 coin flips of fair coins will result in more than $501,000$ heads, using the De Moivre-Laplace Theorem. (Some of the following integrals may be relevant: $\int_{-\infty}^{0}e^{-t^2/2}dt/\sqrt{2\pi}=1/2$, $\int_{-\infty}^{1}e^{-t^2/2}dt/\sqrt{2\pi}\approx .8413$, $\int_{-\infty}^{2}e^{-t^2/2}dt/\sqrt{2\pi}\approx .9772$, $\int_{-\infty}^{3}e^{-t^2/2}dt/\sqrt{2\pi}\approx .9987$.)

Casinos do these kinds of calculations to make sure they make money and that they do not go bankrupt. Financial institutions and insurance companies do similar calculations for similar reasons.

Exercise 6. Find a doubly-infinite array of real numbers $\{a_{ij}\}_{i,j\geq 0}$ such that

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}a_{ij}\right)=1\neq 0=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}a_{ij}\right).$$

(Hint: the array can be chosen to have all entries either $-1, 0$, or 1. And most of the entries can be chosen to be 0.)

Exercise 7. Let X, Y be independent, discrete random variables. Using a total probability theorem-type argument, show that

$$\mathbf{P}(X+Y=z)=\sum_{x\in\mathbf{R}}\mathbf{P}(X=x)\mathbf{P}(Y=z-x), \qquad \forall\, z\in\mathbf{R}.$$

Exercise 8. Let X, Y be independent, continuous random variables with densities f_X, f_Y, respectively. Let f_{X+Y} be the density of $X+Y$. Show that

$$f_{X+Y}(z)=\int_{\mathbf{R}} f_X(x)f_Y(z-x)dx, \qquad \forall z\in\mathbf{R}.$$

Using this identity, find the density f_{X+Y} when X and Y are both independent, uniformly distributed on $[0,1]$.