171 Final Solutions, Winter 2017[1]

1. Question 1

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Classify state 3 as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

Solution. State 3 is transient since $P(3, 2) > 0$, while $P^n(2, 3) = 0$ for all $n \geq 1$. In fact, it follows by induction on n and the definition of P that $P^n(i, j) = 0$ for all $1 \leq i \leq 2$ and $3 \leq j \leq 5$. By the definition of matrix multiplication, by the inductive hypothesis, and by definition of P, we have

$$P^{n+1}(i, j) = \sum_{k=1}^{5} P^n(i, k)P(k, j) = \sum_{k=1}^{2} P^n(i, k)P(k, j) = \sum_{k=1}^{2} P^n(i, k) \cdot 0 = 0.$$

So, if $X_1 = 2$, we know that $X_n \in \{1, 2\}$ for all $n \geq 1$ with probability 1. Therefore,

$$\mathbf{P}_3(T_3 = \infty) \geq \mathbf{P}_3(X_1 = 2, X_n \in \{1, 2\} \, \forall \, n \geq 2) = \mathbf{P}_3(X_1 = 2) = P(3, 2) = 1/3 > 0.$$

The Markov chain is not irreducible, since as we mentioned above, $P^n(1, 3) = 0$ for all $n \geq 1$.

2. Question 2

Consider a non-standard 4×4 chess board. Let V be a set of vertices corresponding to each square on the board (so V has 16 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a knight can move from x to y. (The knight chess piece moves in an L-shape, so that a single move constitutes two spaces moved along the horizontal axis followed by one move along the vertical axis (or two spaces moved along the vertical axis, followed by one move along the horizontal axis.) Consider the simple random walk on this graph. This Markov chain then represents a knight randomly moving around the chess board. For every space x on the chessboard, compute the expected return time $\mathbb{E}_x T_x$ for that space.

When you are done, write $\mathbb{E}_x T_x$ for each point x in the chess board below. (You may assume the Markov chain is irreducible.)

Solution. By Corollary 3.37, if π is the unique solution to $\pi = \pi P$, then $\mathbb{E}_x T_x = 1/\pi(x)$. So, it suffices to find $\pi(x)$ for any $x \in \Omega$. From Example 3.50 in the notes, $\pi(x) = \deg(x)/(2\,|E|)$. (From a previous exercise, we know that $\sum_{x \in V} \deg(x) = 2\,|E|$.) The following table depicts

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the degrees of each entry in the chess board

$$\begin{pmatrix} 2 & 3 & 3 & 2 \\ 3 & 4 & 4 & 3 \\ 3 & 4 & 4 & 3 \\ 2 & 3 & 3 & 2 \end{pmatrix}.$$

So, $48 = \sum_{x\in V} \deg(x) = 2\,|E|$, so $\mathbb{E}_x T_x = 1/\pi(x) = (2\,|E|)/\deg(x) = 48/\deg(x)$. So, the following table depicts $\mathbb{E}_x T_x$ at each point on the chessboard

$$\begin{pmatrix} 24 & 16 & 16 & 24 \\ 16 & 12 & 12 & 16 \\ 16 & 12 & 12 & 16 \\ 24 & 16 & 16 & 24 \end{pmatrix}.$$

3. Question 3

Give an example of a random walk on a graph that is not reversible.

Solution. Let P be any doubly stochastic matrix that is not symmetric, and such that the Markov chain is irreducible. Then the Markov chain will not be reversible. Since P is doublye staochastic, the (unique) stationary distribution is uniform, since

$$(\pi P)(x) = \sum_{y\in\Omega} \pi(y)P(y,x) = \frac{1}{|\Omega|}\sum_{y\in\Omega} P(y,x) = \frac{1}{|\Omega|} = \pi(x).$$

So reversibility reduces to $P(x,y) = P(y,x)$ for all $x, y \in \Omega$. And this equality will not hold when P is not symmetric.

For example, let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The (left) eigenvector of P corresponding to the eigenvalue 1 is $(1, 1, 1)$. So the unique stationary distribution satisfies $\pi(1) = \pi(2) = \pi(3) = 1/3$. But $\pi(1)P(1,2) = (1/3)(1) = 1/3$ whereas $\pi(2)P(2,1) = (1/3)(0) = 0$.

4. Question 4

Suppose we have a finite, irreducible, aperiodic Markov chain with transition matrix P. Since there exists a unique stationary distribution for this Markov chain, we know that one eigenvalue of P is 1.

Show that any other eigenvalue λ of P satisfies $|\lambda| < 1$.

Solution. We argue by contradiction. Suppose P has an eigenvalue λ with $|\lambda| \geq 1$ and $\lambda \neq 1$. Let μ be a (left) eigenvector of P with eigenvalue λ. Then $\mu P = \lambda\mu$, so that $\mu P^n = \lambda^n \mu$ for any $n \geq 1$. The Convergence Theorem (Theorem 3.62 in the notes) implies that as $n \to \infty$, P^n converges to a matrix Π each of whose rows is the stationary distribution π. So,

$$\mu\Pi = \lim_{n\to\infty} \mu P^n = \lim_{n\to\infty} \lambda^n \mu.$$

In particular, the limit on the right exists. This limit can only exist if $|\lambda| \leq 1$ and $\lambda \neq -1$. Since $\lambda \neq 1$, we conclude that $|\lambda| < 1$.

5. Question 5

Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $0 < p < 1$ with $p \neq 1/2$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$.

Compute $\mathbb{E}T$, in terms of a, b, p.

(Hint: use martingales, somehow. If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

(Second hint: you can freely use the formula $\mathbf{P}(Y_T = a) = \frac{(q/p)^{x_0} - (q/p)^b}{(q/p)^a - (q/p)^b}$, where $q := 1 - p$.)

Solution. Let $Z_0 := 0$. For any $n \geq 1$, let $Z_n := Y_n - n\mu$, where $\mu := \mathbb{E}X_1 = p - (1 - p) = 1 - 2p$. As shown in the notes in Example 4.16, $Z_0, Z_1, Z_2, \ldots$ is a martingale. So, from the Optional Stopping Theorem, Version 2,

$$0 = \mathbb{E}Z_0 = \mathbb{E}Z_T = \mathbb{E}Y_T - \mu\mathbb{E}T.$$

That is,

$$\mathbb{E}T = \frac{1}{\mu}\mathbb{E}Y_T = \frac{1}{1 - 2p}(ca + (1 - c)b).$$

6. Question 6

Let $\{N(s)\}_{s \geq 0}$ be a Poisson Process with parameter $\lambda > 0$. Using the **definition of the Poisson process**, show that, for any $s \geq 0$, $N(s)$ is a Poisson random variable with parameter λs.

(Recall, X is a Poisson random variable with parameter $\lambda > 0$ if $\mathbf{P}(X = n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$ for all nonnegative integers n.)

(Hint: a gamma distributed random variable T_n with parameters n and λ has density

$$f_{T_n}(t) := \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, & \text{if } t \geq 0 \\ 0, & \text{otherwise.)} \end{cases}$$

Let n be a nonnegative integer. Then

$$\begin{aligned}
\mathbf{P}(N(s) = n) &= \mathbf{P}(\max\{m \geq 0 \colon T_m \leq s\} = n) = \mathbf{P}(T_n \leq s,\, T_{n+1} > s) \\
&= \mathbf{P}(T_n \leq s,\, T_n + \tau_{n+1} > s) \\
&= \int_{-\infty}^{s} \int_{s-t}^{\infty} f_{\tau_{n+1}}(y) f_{T_n}(t)\, dy dt, \qquad \text{since } T_n \text{ and } \tau_{n+1} \text{ are independent} \\
&= \int_{-\infty}^{s} \mathbf{P}(\tau_{n+1} > s - t) f_{T_n}(t) dt = \int_0^s e^{-\lambda(s-t)} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} dt, \qquad \text{by the } f_{T_n} \text{ formula} \\
&= e^{-\lambda s} \frac{\lambda^n}{(n-1)!} \int_0^s t^{n-1} dt = e^{-\lambda s} \frac{\lambda^n s^n}{n!}.
\end{aligned}$$

7. Question 7

Suppose you run a (busy) car wash, and the number of cars that come to the car wash between time 0 and time $s > 0$ is a Poisson process with rate $\lambda = 1$. Suppose every car has either one, two, three, or four people in it. The probability that a car has one, two, three or four people in it is 1/2, 1/8, 1/8 and 1/4, respectively.

What is the average number of cars with four people that have arrived by time $s = 60$?

Solution. From Theorem 5.17 in the notes, the number of cars with four people in it is a Poisson process with rate $\lambda \cdot (1/4) = 1/4$. So, the average number of cars with four people is the expected value $\mathbb{E}N(60)$ of a Poisson Process with rate $1/4$. From Lemma 5.5 in the notes, $N(60)$ is a Poisson random variable with parameter $60(1/4) = 15$. That is, $\mathbf{P}(N(60) = n) = e^{-15}15^n/n!$ for any nonnegative integer n. So,

$$\mathbb{E}N(60) = e^{-15}\sum_{n=0}^{\infty} n\frac{15^n}{n!} = e^{-15}15\sum_{n=0}^{\infty}\frac{15^n}{n!} = e^{-15}e^{15}15 = 15.$$

8. Question 8

Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbb{Z}$. For any $n \geq 0$, define $M_n = X_n^3 - 3nX_n$. Show that $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$

Now, fix $m > 0$ and let T be the first time that the walk hits either 0 or m. Show that, for any $0 < k \leq m$,

$$\mathbb{E}_k(T \,|\, X_T = m) = \frac{m^2 - k^2}{3}.$$

(You can apply the Optional stopping theorem without verifying that the martingale is bounded.)

Solution.

$$\begin{aligned}
&\mathbb{E}(M_{n+1} - M_n \,|\, X_n = x_n, \ldots, X_0 = x_0, M_0 = m_0)\\
&= \mathbb{E}(([X_{n+1} - X_n] + x_n)^3 - 3(n+1)([X_{n+1} - X_n] + x_n) - x_n^3 + 3nx_n \,|\, X_n = x_n)\\
&= \frac{1}{2}\Big((1 + x_n)^3 - 3(n+1)(1 + x_n) - x_n^3 + 3nx_n\Big)\\
&\qquad\qquad + \frac{1}{2}\Big((-1 + x_n)^3 - 3(n+1)(-1 + x_n) - x_n^3 + 3nx_n\Big)\\
&= \frac{1}{2}\Big((1 + x_n)^3 - x_n^3 - 3x_n + (-1 + x_n)^3 - x_n^3 - 3x_n\Big) = \frac{1}{2}\Big(3x_n^2 + 1 - 3x_n^2 - 1\Big) = 0.
\end{aligned}$$

Now, if $X_0 = k$ with $0 \leq k \leq m$, then the Optional Stopping Theorem says

$$k^3 = \mathbb{E}_k X_0^3 = \mathbb{E}_k M_0 = \mathbb{E}_k M_T = \mathbb{E}_k X_T^3 - 3\mathbb{E}_k T X_T.$$

(Note that $\mathbf{P}(T < \infty) = 1$ by Lemma 3.27 in the notes.) Let $p := \mathbf{P}_k(T = m)$. Since $T = 0$ or $T = m$, the Total Expectation Theorem says

$$\begin{aligned}
\mathbb{E}_k T X_T &= \mathbf{P}_k(T = m)\mathbb{E}_k(T X_T \,|\, T = m) + \mathbf{P}_k(T = 0)\mathbb{E}_k(T X_T \,|\, T = 0)\\
&= mp\mathbb{E}_k(X_T \,|\, T = m).
\end{aligned}$$

Also, $\mathbb{E}_k X_T^3 = m^3 p$. So, we have

$$k^3 = m^3 p - 3mp\mathbb{E}_k(X_T \,|\, T = m).$$

From Example 4.28 in the notes, $p = \frac{k}{m}$. So,

$$\mathbb{E}_k(X_T \,|\, T = m) = \frac{k^3 - m^3 p}{-3mp} = \frac{m^2 k - k^3}{3k} = \frac{m^2 - k^2}{3}.$$

9. Question 9

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Let $x_1, \ldots, x_n \in \mathbb{R}$, and let $t_n > \cdots > t_1 > 0$. Show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad \forall\, x \in \mathbb{R},\, t > 0.$$

Solution.

$$\begin{aligned}&\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}\\ &\qquad = \{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_{n-1}\}.\end{aligned}$$

The random variables listed on the right are all independent, by the independent increment property (i) of Brownian motion. So, the joint density of $(B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}))$ is the product of the respective densities of the random variables. By property (ii) of Brownian motion, $B(s) - B(t)$ is a Gaussian random variable with mean zero and variance $t - s$. So, the joint density of $(B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}))$ has density $f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$. The proof is complete.

10. Question 10

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Given that $B(1) = 10$, what is the expected length of time after $t = 1$ until $B(t)$ hits either 8 or 12?
- Now, let $\sigma = 2$, and $\mu = -5$. Suppose a commodity has price $X(t) = \sigma B(t) + \mu t$ for any time $t \geq 0$. Given that the price of the commodity is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$? (You can leave your final answer here as an integral.)

Solution. Let $T := \inf\{t \geq 1\colon B(t) = 8 \text{ or } B(t) = 12\}$. From the independent increment property, $\{B(t+1) - B(1)\}_{t\geq 0}$ is a standard Brownian motion that is independent of $B(1)$. So, given that $B(1) = 10$, $T := \inf\{t \geq 0\colon B(t+1) - B(1) = -2 \text{ or } B(t+1) - B(1) = 2\}$. So, if $\{Z(t)\}_{t\geq 0}$ is a standard Brownian motion and if $S := \inf\{t \geq 0\colon Z(t) = -2 \text{ or } Z(t) = 2\}$, we have

$$\mathbb{E}(T|B(1) = 10) = \mathbb{E}S.$$

From Proposition 7.13 in the notes, $\mathbb{E}S = 4$. So, $\mathbb{E}(T|B(1) = 10) = 4$.

We now answer the second question. It is given that $X(8) = 4$. That is, $\sigma B(8) + 8\mu = 4$, so $B(8) = (4 - 8\mu)/\sigma$. We want to find the probability that $X(9) < 1$, i.e. $\sigma B(9) + 9\mu < 1$, i.e. $B(9) < (1 - 9\mu)/\sigma$. That is, we want to compute the probability that $B(9) - B(8) + B(8) < (1 - 9\mu)/\sigma$. By the independent increment property, $B(9) - B(8)$ is a standard Gaussian random variable which is independent of $B(8)$. So, we need to compute the probability that

$B(9) - B(8) < (1 - 9\mu)/\sigma + (8\mu - 4)/\sigma$. So, if Y is a standard Gaussian random variable, we need to compute

$$\mathbf{P}\left(Y < \frac{-3-\mu}{\sigma}\right) = \int_{-\infty}^{(-3-\mu)/\sigma} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}.$$