171 Midterm 1 Solutions, Fall 2016[1]

1. Question 1

True/False

(a) Let Ω be a universe. Let $A_1, A_2, \ldots \subseteq \Omega$. Then

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \Omega \colon \forall \text{ positive integers } j, \quad x \in A_j\}.$$

FALSE. If $A_1 = \Omega$ and $A_2 = \emptyset$, then $\bigcup_{i=1}^{\infty} A_i = \Omega$, but $\emptyset = \{x \in \Omega \colon \forall \text{ positive integers } j, \quad x \in A_j\}$.

(b) For any positive integers i, j, let a_{ij} be a real number. Then

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

FALSE. For any $i \geq 1$, let $a_{i(i+1)} = 1$, let $a_{ii} = -1$, and let $a_{ij} = 0$ for any other i, j. Then $\sum_{i=1}^{\infty}\left(a_{ii} + a_{i(i+1)}\right) = \sum_{i=1}^{\infty}(0) = 0 \neq -1 = a_{11} + 0 = a_{11} + \sum_{j=2}^{\infty}\left(a_{(j-1)j} + a_{jj}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$

(c) The Markov Chain with transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ has exactly two recurrent states.

FALSE; All three states are recurrent. Since $P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for any $x \in \{1, 2, 3\}$ we have $\mathbf{P}_x(X_3 = x) = 1$. So, $\mathbf{P}_x(T_x \leq 3) = 1$, and $\mathbf{P}_x(T_x < \infty) = 1$.

2. Question 2

Suppose X and Y are independent standard Gaussian distributed random variables. (So, $\mathbf{P}(a \leq X \leq b) = \int_a^b e^{-t^2/2} dt/\sqrt{2\pi}$, for any $-\infty \leq a \leq b \leq \infty$.) Find the probability density function of $X + Y$. That is, find a function $f_{X+Y} \colon \mathbb{R} \to [0, \infty)$ such that $\mathbf{P}(a \leq X + Y \leq b) = \int_a^b f_{X+Y}(t) dt$ for all $-\infty \leq a \leq b \leq \infty$.

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Solution. Let $t \in \mathbb{R}$. Then

$$\begin{aligned}
f_{X+Y}(t) &= \frac{d}{dt}\mathbf{P}(X+Y<t) = \frac{d}{dt}\iint_{\{x+y<t\}} f_{X,Y}(x,y)dxdy \\
&= \frac{d}{dt}\iint_{\{x+y<t\}} f_X(x)f_Y(y)dxdy, \qquad \text{since } X,Y \text{ are independent} \\
&= \frac{d}{dt}\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=t-y} f_X(x)f_Y(y)dxdy \\
&= \int_{y=-\infty}^{y=\infty} f_X(t-y)f_Y(y)dy, \qquad \text{by the Fundamental Theorem of Calculus} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{y=\infty} e^{-(t-y)^2/2}e^{-y^2/2}dy, \qquad \text{by definition of } X,Y \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{y=\infty} e^{-t^2/2-y^2+ty}dy = \frac{1}{\sqrt{2\pi}}e^{-t^2/4}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{y=\infty} e^{-(y-t/2)^2}dy \\
&= \frac{1}{\sqrt{2\pi}}e^{-t^2/4}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{y=\infty} e^{-y^2}dy \\
&= \frac{1}{2\sqrt{\pi}}e^{-t^2/4}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{y=\infty} e^{-y^2/2}dy, \qquad \text{changing variables} \\
&= \frac{1}{2\sqrt{\pi}}e^{-t^2/4}
\end{aligned}$$

3. Question 3

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1,2,3,4,5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

Classify all states in the Markov chain as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

Solution. State 1 is recurrent, since

$$\mathbf{P}_1(T_1=\infty) = \mathbf{P}_1(2 = X_2 = X_3 = X_4 = \cdots) = \lim_{n\to\infty} P(1,2)(P(2,2))^n = \lim_{n\to\infty}(3/4)(1/4)^n = 0.$$

State 2 is recurrent, since

$$\mathbf{P}_2(T_2=\infty) = \mathbf{P}_2(1 = X_2 = X_3 = X_4 = \cdots) = \lim_{n\to\infty}(P(1,1))^n = \lim_{n\to\infty}(1/4)^n = 0.$$

State 3 is recurrent since $\mathbf{P}_3(T_3 = 1) = 1$, so $\mathbf{P}_3(T_3 < \infty) = 1$.

States 4 and 5 are transient, since

$$\mathbf{P}_4(T_4=\infty) \geq \mathbf{P}_4(3 = X_2 = X_3 = X_4 = \cdots) = P(4,3)\lim_{n\to\infty} P(3,3)^n = P(4,3) > 0.$$

$$\mathbf{P}_5(T_5=\infty) \geq \mathbf{P}_5(3 = X_2 = X_3 = X_4 = \cdots) = P(5,3)\lim_{n\to\infty} P(3,3)^n = P(5,3) > 0.$$

The Markov chain is not irreducible, since P is a block matrix of the form $P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A is 2×2 and B is 3×3. So, for any $n \geq 1$, $P^n = \begin{pmatrix} A^n & 0 \\ 0 & B^n \end{pmatrix}$. So, $P^n(1,3) = 0$ for any $n \geq 1$. So, the Markov chain is not irreducible.

4. Question 4

Let x, y be any states in a finite irreducible Markov chain. Show that $\mathbb{E}_x T_y < \infty$.

Solution. From Lemma 3.27 in the notes, there exists $0 < \alpha < 1$ and $j > 0$ such that, for any $x, y \in \Omega$ and for any $k > 0$, $\mathbf{P}_x(T_y > kj) \leq \alpha^k$. So, $\mathbf{P}_x(T_y > kj) \leq \alpha^k$. So, using Remark 2.23 in the notes,

$$\mathbb{E}_x T_y = \sum_{k=1}^{\infty} \mathbf{P}_x(T_y \geq k) = \sum_{k=1}^{\infty} \sum_{j(k-1) < i \leq jk} \mathbf{P}_x(T_y \geq i)$$
$$\leq \sum_{k=1}^{\infty} j \mathbf{P}_x(T_y > j(k-1)) \leq j \sum_{k=1}^{\infty} \alpha^{k-1} = j/(1-\alpha) < \infty.$$

5. Question 5

Prove or disprove the following statement.

Every finite Markov chain on a nonempty state space has at least one recurrent state.

Solution. This statement is True. We argue by contradiction. Suppose every state is transient. That is, every $y \in \Omega$ satisfies $0 \leq \rho_{yy} < 1$, where $\rho_{yy} = \mathbf{P}_y(T_y < \infty)$. As in the notes, let $T_y^{(1)} := T_y$, and for any $k \geq 2$, define a random variable $T_y^{(k)} := \min\{n > T_y^{(k-1)} \colon X_n = y\}$. That is, $T_y^{(k)}$ is the k^{th} return time of the Markov chain. If $k \geq 1$ is fixed, by the pigeonhole principle, at the $|\Omega| \cdot k^{th}$ step of the Markov chain, the chain has returned to some state k times. That is, for any $k \geq 1$ fixed, there exists $y \in \Omega$ such that $T_y^{(k)} < \infty$ with probability one. That is, for any fixed $k \geq 1$,

$$1 = \mathbf{P}\left(\cup_{y \in \Omega} \{T_y^{(k)} < \infty\}\right).$$

Using the union bound,

$$1 \leq \sum_{y \in \Omega} \mathbf{P}(T_y^{(k)} < \infty).$$

From Proposition 3.21 in the notes, $\mathbf{P}(T_y^{(k)} < \infty) = \rho_{yy}^k$. So,

$$1 \leq \sum_{y \in \Omega} \rho_{yy}^k \leq |\Omega| \, (\max_{y \in \Omega} \rho_{yy})^k.$$

Since Ω is finite, $0 \leq \max_{y \in \Omega} \rho_{yy} < 1$. So, if $k > \frac{\log|\Omega|}{\log(1/\max_{y \in \Omega} \rho_{yy})}$, we have $1 < 1$, a contradiction. The proof is complete.