Math 171, Fall 2016, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ____________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Mid-Term 1

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

| Problem | Points | Score |
|---|---|---|
| 1 | 9 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 49 | |

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**Reference sheet**

Below are some definitions that may be relevant.

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A **finite Markov Chain** is a stochastic process $(X_0, X_1, X_2, \ldots)$ together with a finite set $\Omega$, which is called the **state space** of the Markov Chain, and an $|\Omega| \times |\Omega|$ real matrix $P$. The random variables $X_0, X_1, \ldots$ take values in the finite set $\Omega$. The matrix $P$ is **stochastic**, that is all of its entries are nonnegative and

$$\sum_{y \in \Omega} P(x, y) = 1, \qquad \forall\, x \in \Omega.$$

And the stochastic process satisfies the following **Markov property**: for all $x, y \in \Omega$, for any $n \geq 1$, and for all events $H_{n-1}$ of the form $H_{n-1} = \cap_{k=0}^{n-1}\{X_k = x_k\}$, where $x_k \in \Omega$ for all $0 \leq k \leq n-1$, such that $\mathbf{P}(H_{n-1} \cap \{X_n = x\}) > 0$, we have

$$\mathbf{P}(X_{n+1} = y \,|\, H_{n-1} \cap \{X_n = x\}) = \mathbf{P}(X_{n+1} = y \,|\, X_n = x) = P(x, y).$$

Suppose we have a Markov Chain $X_0, X_1, \ldots$ with state space $\Omega$. Let $y \in \Omega$. Define the **first return time** of $y$ to be the following random variable: $T_y := \min\{n \geq 1 \colon X_n = y\}$. Also, define $\rho_{yy} := \mathbf{P}_y(T_y < \infty)$.

If $\rho_{yy} = 1$, we say the state $y \in \Omega$ is **recurrent**. If $\rho_{yy} < 1$, we say the state $y \in \Omega$ is **transient**.

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Let $\Omega$ be a universe. Let $A_1, A_2, \ldots \subseteq \Omega$. Then

   $$\bigcup_{i=1}^{\infty} A_i = \{x \in \Omega \colon \forall \text{ positive integers } j, \quad x \in A_j\}.$$

   TRUE FALSE (circle one)

   (b) (3 points) For any positive integers $i, j$, let $a_{ij}$ be a real number. Then

   $$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

   TRUE FALSE (circle one)

   (c) (3 points) The Markov Chain with transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ has exactly two recurrent states.

   TRUE FALSE (circle one)

2. (10 points) Suppose $X$ and $Y$ are independent standard Gaussian distributed random variables. (So, $\mathbf{P}(a \leq X \leq b) = \int_a^b e^{-t^2/2}dt/\sqrt{2\pi}$, for any $-\infty \leq a \leq b \leq \infty$.) Find the probability density function of $X+Y$. That is, find the function $f_{X+Y}\colon \mathbf{R} \to [0,\infty)$ such that

$$\mathbf{P}(a \leq X+Y \leq b) = \int_a^b f_{X+Y}(t)dt, \qquad \forall -\infty \leq a \leq b \leq \infty$$

3. (10 points) Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

Classify all states in the Markov chain as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

4. (10 points) Let $x, y$ be any states in a finite irreducible Markov chain. Show that

$$\mathbf{E}_x T_y < \infty.$$

5. (10 points) Prove or disprove the following statement.

   Every finite Markov chain on a nonempty state space has at least one recurrent state.

(Scratch paper)