Please provide complete and well-written solutions to the following exercises.

Due March 16, in the discussion section.

Homework 8

Exercise 1. Let $T_1, T_2, \ldots$ be independent geometric random variables with parameter p. For any integer $k \geq 1$, let $Y_k := T_1 + \cdots + T_k$. Show that the PMF of Y_k is given by

$$p_{Y_k}(t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k} & \text{, if } t \geq k,\ t \in \mathbb{Z} \\ 0 & \text{, otherwise.} \end{cases}$$

Exercise 2. Give an alternate proof that $\mathbf{P}(X_{k+1} = 1) = p$ in the Proposition (An Equivalent Definition of Bernoulli Process) by using the following conditioning argument:

$$\mathbf{P}(X_{k+1} = 1) = \sum_{n=1}^{k+1} \mathbf{P}(X_{k+1} = 1 \,|\, T_1 = n) \mathbf{P}(T_1 = n)$$

$$= \mathbf{P}(X_{k+1} = 1 \,|\, T_1 = k+1) \mathbf{P}(T_1 = k+1) + \sum_{n=1}^{k} \mathbf{P}(X_{k+1} = 1 \,|\, T_1 = n) \mathbf{P}(T_1 = n)$$

$$= \mathbf{P}(T_1 = k+1) + \sum_{n=1}^{k} \mathbf{P}(T_1 + \cdots + T_j = k+1 \text{ for some } j \geq 2 \,|\, T_1 = n) \mathbf{P}(T_1 = n) = \cdots$$

Exercise 3. Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. What is the expected number of trials that have to occur before we see two consecutive "successes"?

Exercise 4. Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. Define $N := \min\{n \geq 1 \colon X_n \neq X_1\}$. For any $n \geq 1$, define $Y_n := X_{N+n-2}$. Show that $\mathbf{P}(Y_n = 1) = 1/2$ for all $n \geq 1$, but $Y_1, Y_2, \ldots$ is not a Bernoulli process.

Exercise 5. Suppose the number of students going to a restaurant in Ackerman in a single day has a Poisson distribution with mean 500. Suppose each student spends an average of \$10 with a standard deviation of \$5. What is the average revenue of the restaurant in one day? What is the standard deviation of the revenue in one day? (The amounts spent by the students are independent identically distributed random variables.)

Exercise 6. Let $X_0 := 0$. Let $X_0, X_1, \ldots$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$ for all $n \geq 1$. Let $S_0, S_1, \ldots$ be the corresponding random walk started at 0. Let $T := \min\{n \geq 1 \colon S_n = 1\}$. Show that T is a stopping time.

Exercise 7. Let $X_0 := x_0 \in \mathbb{Z}$. Let $X_0, X_1, \ldots$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$ for all $n \geq 1$. Let $S_0, S_1, \ldots$ be the corresponding random walk started at x_0. Let $a, b \in \mathbb{Z}$ such that $a < x_0 < b$. Let $T := \min\{n \geq 1 \colon S_n \in \{a, b\}\}$. Show that T is a stopping time.