170B Final Solutions, Winter 2017[1]

1. Question 1

True/False

(a) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω. Let $\mathbf{P}$ denote a probability law on Ω. Then

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right)$$

FALSE. Let $A_1 = A_2 = \Omega$ and let $\emptyset = A_3 = A_4 = \cdots$. Then the left side is $1 + 1 = 2$, but the right side is $\mathbf{P}(\Omega) = 1$.

(b) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbb{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

FALSE. Let $f_X(t) := 1$ for any $t \in [0, 1]$ and let $f_X(t) := 0$ otherwise. Then

$$\mathbf{P}(X \leq t) = \begin{cases} 0 & , \text{if}\, t < 0 \\ t & , \text{if}\, 0 \leq t \leq 1 \\ 1 & , \text{if}\, t > 1 \end{cases}.$$

In particular, $\frac{d}{dt}\mathbf{P}(X < t)$ does not exist at $t = 0$.

(c) Let X be a random variable such that $\mathbb{E}X^4 < \infty$. Then $\mathbb{E}X^2 < \infty$.

TRUE. By Jensen's inequality, $(\mathbb{E}X^2)^2 \leq \mathbb{E}X^4 < \infty$.

(d) Let X be a random variable such that $\mathrm{var}(X) = 2$. Then $\mathbf{P}(|X - \mathbb{E}X| > 2) \leq 1/2$.

TRUE. By Chebyshev's inequality,

$$\mathbf{P}(|X - \mathbb{E}X| > 2) \leq \mathrm{var}(X)/4 = 1/2.$$

(e) Let $i = \sqrt{-1}$. Let $X_1, X_2, \ldots$ be random variables such that, for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}e^{itX_n} = e^{-t^2/2}.$$

Then $X_1, X_2, \ldots$ converges in distribution to a standard Gaussian random variable.

TRUE. This is basically how we proved the Central Limit Theorem (Theorem 3.21 in the notes). This assertion follows by the Levy Continuity Theorem, and using that $\mathbb{E}e^{itZ} = e^{-t^2/2}$ for all $t \in \mathbb{R}$ where Z is a standard Gaussian random variable (Prop. 2.55 in the notes).

(f) Let X be a standard Gaussian random variable (so that $\mathbf{P}(X \leq t) = \int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}$ for any $t \in \mathbb{R}$.) Then

$$\mathbf{P}(X > t) < \frac{1}{t}, \quad \forall\, t > 0$$

TRUE. This follows from Markov's inequality. First, note that $\mathbb{E}\,|X| = 2\int_0^\infty xe^{-x^2/2}\frac{dx}{\sqrt{2\pi}} = \sqrt{2/\pi} < 1$. So, using Markov's inequality,

$$\mathbf{P}(X > t) \leq \mathbf{P}(|X| > t) \leq \frac{\mathbb{E}\,|X|}{t} < \frac{1}{t}.$$

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(g) Let $\{N(s)\}_{s\geq 0}$ be a Poisson Process with parameter $\lambda = 1$. Then

$$\mathbb{E}((N(4)-N(2))N(2)) = 4.$$

TRUE. $\mathbb{E}((N(4)-N(2))N(2)) = \mathbb{E}((N(4)-N(2)))\mathbb{E}N(2) = [\mathbb{E}N(2)]^2 = 2^2 = 4$. Here we used the independent increment property (that $N(4)-N(2)$ is independent of $N(2)-N(0) = N(2)$), and that $N(2)$ is a Poisson random variable with parameter $\lambda \cdot 2 = 2$, so that $\mathbb{E}N(2) = 2$.

(h) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then $X_1, X_2, \ldots$ converges almost surely to X.

FALSE. Let $\Omega = [0,1]$. For any $n \geq 1$, let

$$X_n(\omega) := \begin{cases} (-1)^n & \text{, if } \omega \in [0,1/2) \\ (-1)^{n+1} & \text{, if } \omega \in [1/2,1]. \end{cases}$$

Then $X_1, X_2, \ldots$ all have the same distribution, so they converge in distribution, but they do not converge almost surely, since $\lim_{n\to\infty} X_n(\omega)$ does not exist for every $\omega \in [0,1]$.

(i) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$$

FALSE. Let $\Omega = [0,1]$. For any $n \geq 1$, let

$$X_n(\omega) := \begin{cases} n & \text{, if } \omega \in [0,1/n] \\ 0 & \text{, if } \omega \in (1/n,1]. \end{cases}$$

Then $\mathbb{E}X_n = 1$ for all $n \geq 1$, but $X_1, X_2, \ldots$ converges in probability to 0 as $n \to \infty$, as shown in class. So, $\lim_{n\to\infty} \mathbb{E}X_n = 1 \neq 0 = \mathbb{E}X$.

2. Question 2

Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$, and such that $\cup_{i=1}^n C_i = B$. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^n \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B).$$

Solution. From the Total Probability Theorem applied to $\mathbf{P}(\cdot|B)$, and then using the definition of conditional probability,

$$\begin{aligned} \mathbf{P}(A|B) &= \sum_{i=1}^n \mathbf{P}(A\cap C_i|B) = \sum_{i=1}^n \frac{\mathbf{P}(A\cap B\cap C_i)}{\mathbf{P}(B)} \\ &= \sum_{i=1}^n \frac{\mathbf{P}(A\cap B\cap C_i)}{\mathbf{P}(B\cap C_i)}\frac{\mathbf{P}(B\cap C_i)}{\mathbf{P}(B)} = \sum_{i=1}^n \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B). \end{aligned}$$

3. Question 3

Let X be a random variable that is uniformly distributed in $[0,1]$. Let $Y := 4X(1-X)$. Find f_Y, the density function of Y.

Solution. Using the quadratic formula, the function $f(t) = 4t(1-t)$ takes the value $c \in [0,1]$ when $x = (1/2) \pm (1/2)\sqrt{1-c}$. So, if $x \in [0,1]$, we have

$$\begin{aligned}\mathbf{P}(4X(1-X) \leq x) &= \mathbf{P}(X \in [0, 1/2 - (1/2)\sqrt{1-x}] \text{ or } X \in [1/2 + (1/2)\sqrt{1-x}, 1]) \\ &= (1/2) - (1/2)\sqrt{1-x} + 1 - (1/2 + (1/2)\sqrt{1-x}) = 1 - \sqrt{1-x}.\end{aligned}$$

We differentiate the CDF to find the density. Then if $0 \leq x \leq 1$, we have

$$f_Y(x) = \frac{d}{dx}(1 - \sqrt{1-x}) = \frac{1}{2}(1-x)^{-1/2}.$$

4. Question 4

Let X, Y be independent random variables. Suppose X has moment generating function

$$M_X(t) = 1 + t^6, \qquad \forall\, t \in \mathbb{R}.$$

Suppose Y has moment generating function

$$M_Y(t) = 1 + t^2, \qquad \forall\, t \in \mathbb{R}.$$

Compute $\mathbb{E}\Big[(X+Y)^2\Big]$.

Solution 1. Since X, Y are independent, we have $M_{X+Y}(t) = M_X(t)M_Y(t) = (1+t^6)(1+t^2) = 1 + t^2 + t^6 + t^8$ for all $t \in \mathbb{R}$ by Proposition 2.43 in the notes. As mentioned in the notes,

$$\mathbb{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0} M_{X+Y}(t).$$

Therefore,

$$\mathbb{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0}\Big(1 + t^2 + t^6 + t^8\Big) = [2 + 30t^4 + 56t^6]_{t=0} = 2.$$

Solution 2. As mentioned in the notes,

$$\mathbb{E}X = \frac{d}{dt}|_{t=0} M_X(t) = \frac{d}{dt}|_{t=0}(1+t^6) = [6t^5]_{t=0} = 0.$$

$$\mathbb{E}X^2 = \frac{d^2}{dt^2}|_{t=0} M_X(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^6) = [30t^4]_{t=0} = 0.$$

$$\mathbb{E}Y = \frac{d}{dt}|_{t=0} M_Y(t) = \frac{d}{dt}|_{t=0}(1+t^2) = [2t]_{t=0} = 0.$$

$$\mathbb{E}Y^2 = \frac{d^2}{dt^2}|_{t=0} M_Y(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^2) = 2.$$

Therefore, using also that X, Y are independent,

$$\mathbb{E}(X+Y)^2 = \mathbb{E}X^2 + \mathbb{E}Y^2 + 2\mathbb{E}(XY) = 0 + 2 + (\mathbb{E}X)(\mathbb{E}Y) = 2 + 0 \cdot 0 = 2.$$

5. Question 5

Using the Central Limit Theorem, prove the Weak Law of Large Numbers.

(You may assume that $X_1, X_2, \ldots$ are independent, identically distributed random variables such that $\mathbb{E}|X_1| < \infty$ and $0 < \mathrm{var}(X_1) < \infty$.)

Solution. Let $\varepsilon > 0$. Let $\sigma := \sqrt{\mathrm{var}(X_1)}$. Then

$$\begin{aligned}\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) &= \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{n}\right| > \varepsilon\right)\\ &= \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > \frac{\sqrt{n}\varepsilon}{\sigma}\right)\end{aligned}$$

So, for any $N > 0$, there exists $m > 0$ such that, for all $n > m$, we have

$$\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) \leq \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > N\right).$$

Letting $n \to \infty$ and using the Central Limit Theorem,

$$\begin{aligned}0 \leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right)\\ &\leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > N\right)\\ &= 2\int_N^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.\end{aligned}$$

The left side does not depend on N, so we let $N \to \infty$ to conclude that

$$0 \leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) \leq \lim_{N\to\infty} 2\int_N^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} = 0.$$

6. Question 6

Suppose you flip a fair coin 120 times. During each coin flip, this coin has probability 1/2 of landing heads, and probability 1/2 of landing tails.

Let A be the event that you get more than 90 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{60}.$$

Solution 1. For any $n \geq 1$, define X_n so that

$$X_n = \begin{cases} 1 & \text{, if the } n^{th} \text{ coin flip is heads} \\ 0 & \text{, if the } n^{th} \text{ coin flip is tails.}\end{cases}$$

By its definition $\mathbb{E}X_n = 1/2$ and $\mathrm{var}(X_n) = (1/2)(1/4) + (1/2)(1/4) = 1/4$.

Let $S := X_1 + \cdots + X_{120}$ be the number of heads that are flipped. Then $\mathbb{E}S = 60$, and $\mathrm{var}(S) = 120\mathrm{var}(X_1) = 30$. Markov's inequality says, for any $t > 0$

$$\mathbf{P}(S > t) \leq \mathbb{E}S/t = 60/t.$$

This is not helpful. Instead, we use Chebyshev's inequality. This says, for any $t > 0$,

$$\mathbf{P}(|S - 60| > t) \leq t^{-2}\mathrm{var}(S) = 30t^{-2}.$$

Choosing $t = 30$ shows that $\mathbf{P}(|S-60| > 10) \leq 1/30$. Now, using symmetry of S (interchanging the roles of heads and tails),

$$\mathbf{P}(|S-60| > 30) = \mathbf{P}(S < 30) + \mathbf{P}(S > 90) = 2\mathbf{P}(S > 90).$$

So,

$$2\mathbf{P}(S > 90) = \mathbf{P}(|S-60| > 30) \leq 1/30.$$

Solution 2. We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Chernoff bound. Since S is a sum of 120 independent identically distributed random variables, Proposition 2.43 from the notes says

$$M_S(t) = (M_{X_1}(t))^{120}, \qquad \forall\, t \in \mathbb{R}.$$

So, the Chernoff bound says, for any $r, t > 0$,

$$\mathbf{P}(S > r) \leq e^{-tr}(M_{X_1}(t))^{120} = e^{-tr}((1/2)(1+e^t))^{120} \qquad (*).$$

Setting $f(t) = e^{-rt}(1+e^t)^{120}$ and solving $f'(t) = 0$ for t shows that $t = \log(3)$ minimizes the quantity $f(t)$. So, choosing $r = 90$ and $t = \log(3)$ in $(*)$ gives

$$\mathbf{P}(S > 90) \leq e^{-tr}((1/2)(1+4))^{120} = (3)^{-90}(5/2)^{120} \leq .0006 < 1/60.$$

Solution 3. (The following solution based on the Central Limit Theorem only received partial credit, since it only approximately shows that $\mathbf{P}(A) < 1/10$.) We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Central Limit Theorem. Since $X_1, X_2, \ldots$ are independent identically distributed random variables with mean $1/2$ and variance $1/4$, the Central Limit Theorem implies that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n/2}{\sqrt{(1/4)}\sqrt{n}} > t\right) = \int_t^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

So, choosing $n = 120$ and $t = \sqrt{30}$, we have the approximation

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_{120} - 60}{\sqrt{(1/4)}\sqrt{120}} > \sqrt{30}\right) \approx \int_{\sqrt{30}}^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

Simplifying a bit,

$$\mathbf{P}\left(S - 60 > 30\right) \approx \int_{\sqrt{30}}^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

Using $\sqrt{30} > 3$ and the approximation $\int_3^\infty e^{-x^2/2}dx/\sqrt{2\pi} \approx .0014$, we have

$$\mathbf{P}(S > 50) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2}dx/\sqrt{2\pi} \leq \int_2^\infty e^{-x^2/2}dx/\sqrt{2\pi} \approx .0014 < 1/60.$$

7. Question 7

Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. What is the expected number of trials that have to occur before we see two consecutive "successes"?

(Your final answer can be left as an infinite sum of numbers. You get three bonus points if your final answer is a single real number that is justified correctly.)

Solution 1. Let T be the number of coin flips that occur until two successive heads occur. From the Total Expectation Theorem,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}(T|X_1=0)\mathbf{P}(X_1=0) + \mathbb{E}(T|X_1=1,X_2=0)\mathbf{P}(X_1=1,X_2=0)\\ &\quad + \mathbb{E}(T|X_1=1,X_2=1)\mathbf{P}(X_1=1,X_2=1)\\ &= \frac{1}{2}\mathbb{E}(T|X_1=0) + \frac{1}{4}\mathbb{E}(T|X_1=1,X_2=0) + \frac{1}{4}\mathbb{E}(T|X_1=1,X_2=1).\end{aligned}$$

From the fresh-start property (or Markov property) of the Bernoulli process, $X_1, X_2, \ldots$ is also a Bernoulli process. That is, if we condition on $X_1 = 0$, then $\mathbb{E}(T|X_1=0) = 1 + \mathbb{E}T$. Similarly, $\mathbb{E}(T|X_1=1,X_2=0) = 2 + \mathbb{E}T$. Also, $\mathbb{E}(T|X_1=1,X_2=1) = 2$, since both successes occurred during the first two coin flips in this case. In summary,

$$\mathbb{E}T = \frac{1}{2}(1+\mathbb{E}T) + \frac{1}{4}(2+\mathbb{E}T) + \frac{1}{4}(2).$$

Rearranging, we get

$$\frac{1}{4}\mathbb{E}T = \frac{3}{2}.$$

That is, $\mathbb{E}T = 6$.

Solution 2. Let T_1 be the number of coin flips that occur until the first success occurs. For any $i \geq 2$, let T_i be the number of coin flips that occur between the i^{th} success and the $(i-1)^{st}$ success. Then the event that two consecutive heads occurs can be written as the disjoint union

$$\cup_{j=2}^{\infty}\{T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j\}.$$

Let T be the number of coin flips that occur until two successive heads occur. Then, by the Total Expectation Theorem, we have

$$\begin{aligned}\mathbb{E}T &= \sum_{j=2}^{\infty}\mathbb{E}(T\,|\,T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\mathbf{P}(T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\\ &= \sum_{j=2}^{\infty}\mathbb{E}(\sum_{k=1}^{j}T_k\,|\,T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\mathbf{P}(T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\end{aligned}$$

From the notes, we know that $T_1, T_2, \ldots$ are independent geometric random variables with parameter $p = 1/2$. Therefore,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}T_1 + \sum_{j=2}^{\infty}\mathbb{E}(\sum_{k=2}^{j}T_k\,|\,T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\mathbf{P}(T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\\ &= \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}\mathbb{E}(\sum_{k=2}^{j-1}T_k\,|\,T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\mathbf{P}(T_j=1,\,T_i>1,\,\forall\,2\leq i<j)\\ &= \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}(j-3)\mathbb{E}(T_1\,|\,T_1>1)2^{-(j-2)} = \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}(j-3)(1+\mathbb{E}T_1)2^{-(j-2)}\\ &= \mathbb{E}T_1 + 1 + (1+\mathbb{E}T_1)\sum_{j=1}^{\infty}(j-1)2^{-j} = \mathbb{E}T_1 + 1 + (1+\mathbb{E}T_1)(\mathbb{E}T_1 - 1) = 2 + 1 + (3)(1) = 6.\end{aligned}$$

8. Question 8

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables such that, for any $i \geq 1$

$$\mathbf{P}(X_i \leq t) = \begin{cases} \frac{t}{t+1} & , \text{if}\, t \geq 0. \\ 0 & , \text{if}\, t < 0. \end{cases}$$

For any $n \geq 1$, let $M_n := \max(X_1, \ldots, X_n)$.

(i) Explicitly compute $\mathbf{P}(M_n \leq t)$ for any $t \in \mathbb{R}$.
(ii) Show that $\frac{M_n}{n}$ converges in distribution to some random variable W, as $n \to \infty$.
(iii) Explicitly compute $\mathbf{P}(1/W \geq t)$ for any $t \in \mathbb{R}$.

Solution. For any $t > 0$, $\{M_n \leq t\} = \{X_1 \leq t, \ldots, X_n \leq t\}$. Using this equality and independence,

$$\mathbf{P}(M_n \leq t) = \prod_{i=1}^{n} \mathbf{P}(X_i \leq t) = (\mathbf{P}(X_1) \leq t)^n = \left(1 - \frac{1}{t+1}\right)^n.$$

And if $t < 0$, then $\mathbf{P}(M_n \leq t) = 0$. Now,

$$\mathbf{P}(M_n/n \leq t) = \mathbf{P}(M_n \leq tn) = \left(1 - \frac{1}{tn+1}\right)^n$$

So, using the power series expansion of the logarithm (or the definition of e), and letting $t > 0$

$$\lim_{n\to\infty} \mathbf{P}(M_n/n \leq t) = \lim_{n\to\infty} \left(1 - \frac{1}{tn+1}\right)^n = e^{-1/t}.$$

So, if we define a random variable W such that $\mathbf{P}(W \leq t) = e^{-1/t}$ for any $t > 0$, and $\mathbf{P}(W \leq t) = 0$ for any $t \leq 0$, then M_n/n converges to W in distribution, as $n \to \infty$. Finally, if $t > 0$, then

$$\mathbf{P}(1/W \geq t) = \mathbf{P}(W \leq \frac{1}{t}) = e^{-t}.$$

That is, $1/W$ is an exponential random variable with parameter 1.

9. Question 9

Let $t_1, t_2, \ldots$ be positive, independent identically distributed random variables. Let $\mu \in \mathbb{R}$. Assume $\mathbb{E}t_1 = \mu$. For any $n \geq 1$, let $T_n := t_1 + \cdots + t_n$. For any positive integer t, let $N_t := \min\{n \geq 1 \colon T_n \geq t\}$.

Show that N_t/t converges almost surely to $1/\mu$ as $t \to \infty$.

(Hint: if c, t are positive integers, then $\{N_t \leq ct\} = \{T_{ct} \geq t\}$. Apply the Strong Law to T_{ct}.)

Solution. From the Strong Law of Large Numbers,

$$\mathbf{P}(\lim_{t\to\infty} T_t/t = \mu) = 1. \qquad (*)$$

By the definition of N_t, we have for any $t \geq 1$,

$$T_{N_t - 1} < t \leq T_{N_t}.$$

Dividing by $N_t > 0$, we get

$$\frac{T_{N_t-1}}{N_t-1}\frac{N_t-1}{N_t} \leq \frac{t}{N_t} < \frac{T_{N_t}}{N_t}. \qquad (**)$$

Also by definition of N_t, for any fixed integer $m > 0$, we have $\mathbf{P}(N_t < m) = \mathbf{P}(T_m > t) \leq \mathbb{E}T_m/t = m\mu/t \to 0$ as $t \to \infty$. So, using this fact and $(*)$, the left and right sides of $(**)$ converge to μ with probability 1. The Theorem follows.

10. Question 10

Let $X_0 := x_0 \in \mathbb{Z}$. Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$ for all $n \geq 1$. Let $S_0, S_1, \ldots$ be the corresponding random walk started at x_0. Let $a, b \in \mathbb{Z}$ such that $a < x_0 < b$. Let $T := \min\{n \geq 1\colon S_n \in \{a, b\}\}$. Show:

$$\mathbf{P}(S_T = a) = \frac{x_0 - b}{a - b}.$$

(You may assume that $\mathbf{P}(T < \infty) = 1$.)

Solution. We claim that T is a stopping time. For any positive integer n,

$$\{T = n\} = \{X_0 \in \{a, b\}^c, \ldots, X_{n-1} \in \{a, b\}^c, X_n \in \{a, b\}\}.$$

Also, $|S_{n\wedge T}| \leq \max(|a|, |b|)$, so the Optional Stopping Theorem, Version 2, applies. Let $c := \mathbf{P}(S_T = a)$. Then

$$x_0 = \mathbb{E}S_0 = \mathbb{E}S_T = ac + (1 - c)b.$$

Solving for c, we get

$$c = \frac{x_0 - b}{a - b}.$$