Please provide complete and well-written solutions to the following exercises.

Due April 11, in the discussion section.

Homework 1

Exercise 1. Let A, B, C be sets in a universe Ω. Using the definitions of intersection, union and complement, prove properties (ii) and (iii) below.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(iii) $(A^c)^c = A$.

(Hint: to prove property (ii), it may be helpful to first draw a Venn diagram of A, B, C. Now, let $x \in \Omega$. Consider where x could possibly be with respect to A, B, C. For example, we could have $x \in A$, $x \notin B$, $x \in C$. We could also have $x \in A$, $x \in B$, $x \notin C$. And so on. In total, there should be $2^3 = 8$ possibilities for the location of x, with respect to A, B, C. Construct a truth table which considers all eight such possibilities for each side of the purported equality $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.)

Exercise 2. Let $A_1, A_2, \ldots$ be sets in some universe Ω. Prove that $\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$.

Exercise 3. Let $A_1, A_2, \ldots$ be sets in some universe Ω. Let $B \subseteq \Omega$. Prove:

$$B \cap \left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} (A_k \cap B).$$

Exercise 4 (Discrete Uniform Probability Law)**.** Let n be a positive integer. Suppose we are given a finite universe Ω with exactly n elements. Let $A \subseteq \Omega$. Define $\mathbf{P}(A)$ such that $\mathbf{P}(A)$ is the number of elements of A, divided by n. Verify that $\mathbf{P}$ is a probability law. This probability law is referred to as the uniform probability law on Ω, since each element of Ω has the same probability.

Exercise 5. Let $\Omega = \mathbf{R}^2$. Let $A \subseteq \Omega$. Define a probability law $\mathbf{P}$ on Ω so that

$$\mathbf{P}(A) = \frac{1}{2\pi} \iint_A e^{-(x^2+y^2)/2} dx dy.$$

We can think of $\mathbf{P}$ as defining the (random) position of a dart, thrown at an infinite dart board. That is, if $A \subseteq \Omega$, then $\mathbf{P}(A)$ is the probability that the dart will land in the set A.

Verify that Axiom (iii) holds for $\mathbf{P}$. That is, verify that $\mathbf{P}(\Omega) = 1$. Then, compute the probability that a dart hits a circular board A, where $A = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$.

Exercise 6. Let A, B be subsets of a sample space Ω. Prove the following things:

- $A = (A \setminus B) \cup (A \cap B)$, and $(A \setminus B) \cap (A \cap B) = \emptyset$.

- $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$, and the three sets $(A \setminus B), (B \setminus A), (A \cap B)$ are all disjoint. That is, any two of these sets are disjoint.

Exercise 7. Let Ω be a sample space and let $\mathbf{P}$ be a probability law on Ω. Let $A, B, C \subseteq \Omega$. Prove the following things:

- $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$.
- $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$.

(**Although the book suggests otherwise, a Venn diagram alone is not a rigorous proof.** As in Exercise 1, a truth table allows us to rigorously reason about the information contained in a Venn diagram. Though, there are ways to do the problem while not directly using a truth table.)

Exercise 8. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function. Show that

$$\cup_{y \in \mathbf{R}} \{x \in \mathbf{R} \colon f(x) = y\} = \mathbf{R}.$$

Also, show that the union on the left is disjoint. That is, if $y_1 \neq y_2$ and $y_1, y_2 \in \mathbf{R}$, then $\{x \in \mathbf{R} \colon f(x) = y_1\} \cap \{x \in \mathbf{R} \colon f(x) = y_2\} = \emptyset$.