Please provide complete and well-written solutions to the following exercises.

Due February 18, in the discussion section.

Homework 6

Exercise 1. Suppose there are ten separate bins. You first randomly place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs twenty times, so that twenty spheres have been placed in bins. What is the expected number of empty bins at the end?

Exercise 2. You want to complete a set of 100 baseball cards. Cards are sold in packs of ten. Assume that each individual card in the pack has a uniformly random chance of being any element in the full set of 100 baseball cards. (In particular, there is a chance of getting identical cards in the same pack.) How many packs of cards should you buy in order to get a complete set of cards? That is, what is the expected number of packs of cards you should buy in order to get a complete set of cards? (Hint: First, just forget about the packs of cards, and just think about buying one card at a time. Let N be the number of cards you need to buy in order to get a full set of cards, so that N is a random variable. More generally, for any $1 \leq i \leq 100$, let N_i be the number of cards you need to buy such that you have exactly i distinct cards in your collection (and before buying the last card, you only had $i-1$ distinct cards in your collection). Note that $N_1 = 1$. Define $N_0 = 0$. Then $N = N_{100} = \sum_{i=1}^{100}(N_i - N_{i-1})$. You are required to compute $\mathbf{E}N$. You should be able to compute $\mathbf{E}[N_i - N_{i-1}]$. This is the expected number of additional cards you need to buy after having already collected $i-1$ distinct cards, in order to see your i^{th} new card.)

Exercise 3. Suppose we are drawing cards out of a standard 52 card deck without replacing them. How many cards should we expect to draw out of the deck before we find (a) a King? (b) a Heart?

Exercise 4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be twice differentiable function. Assume that f is convex. That is, $f''(x) \geq 0$, or equivalently, the graph of f lies above any of its tangent lines. That is, for any $x, y \in \mathbf{R}$,

$$f(x) \geq f(y) + f'(y)(x-y).$$

(In Calculus class, you may have referred to these functions as "concave up.") Let X be a discrete random variable. By setting $y = \mathbf{E}(X)$, prove **Jensen's inequality**:

$$\mathbf{E}f(X) \geq f(\mathbf{E}(X)).$$

In particular, choosing $f(x) = x^2$, we have $\mathbf{E}(X^2) \geq (\mathbf{E}(X))^2$.

Exercise 5. Let n be a positive integer, and let $0 < p < 1$. Let $\Omega = \{0,1\}^n$. Any $\omega \in \Omega$ can then be written as $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_i \in \{0,1\}$ for each $i \in \{1, \ldots, n\}$. Let $\mathbf{P}$ be the probability law so that, for any $\omega \in \Omega$, we have

$$\mathbf{P}(\omega) = \prod_{i=1}^{n} p^{\omega_i}(1-p)^{1-\omega_i} = p^{\sum_{i=1}^{n}\omega_i}(1-p)^{n-\sum_{i=1}^{n}\omega_i}.$$

For each $i \in \{1, \ldots, n\}$, define $X_i \colon \Omega \to \mathbf{R}$ so that $X_i(\omega) = \omega_i$ for any $\omega \in \Omega$. That is, if Ω and $\mathbf{P}$ model the flipping of n distinct biased coins, then $X_i = 1$ when the i^{th} coin is heads, and $X_i = 0$ when the i^{th} coin is tails.

First, show that $\mathbf{P}(\Omega) = 1$. Then, compute the expected value of X_i for each $i \in \{1, \ldots, n\}$. Next, compute the expected value of $Y = \sum_{i=1}^{n} X_i$. Finally, prove that Y is a binomial random variable with parameters n and p.

Exercise 6 (Inclusion-Exclusion Formula)**.** This Exercise gives an alternate proof of the following identity, which is known as the Inclusion-Exclusion Formula: Let $A_1, \ldots, A_n \subseteq \Omega$. Then:

$$\mathbf{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbf{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbf{P}(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} \mathbf{P}(A_i \cap A_j \cap A_k)$$
$$\cdots + (-1)^{n+1}\mathbf{P}(A_1 \cap \cdots \cap A_n).$$

Let Y be a random variable such that $Y = 1$ on $\cup_{i=1}^{n} A_i$, and such that $Y = 0$ otherwise. (That is, $Y(\omega) = 1$ for any $\omega \in \cup_{i=1}^{n} A_i$, and $Y(\omega) = 0$ for any other $\omega \in \Omega$.) For any $i \in \{1, \ldots, n\}$, let X_i be a random variable such that $X_i = 1$ on A_i, and $X_i = 0$ otherwise.

- Show that $Y = 1 - \prod_{i=1}^{n}(1 - X_i)$.
- Expand out the product in the previous item, and take the expected value of both sides of the result. Deduce the Inclusion-Exclusion formula.

Exercise 7. You are trapped in a maze. Your starting point is a room with three doors. The first door will lead you to a corridor which lets you exit the maze after three hours of walking. The second door leads you through a corridor which puts you back to the starting point of the maze after seven hours of walking. The third door leads you through a corridor which puts you back to the starting point of the maze after nine hours of walking. Each time you are at the starting point, you choose one of the three doors with equal probability.

Let X be the number of hours it takes for you to exit the maze. Let Y be the number of the door that you initially choose.

- Compute $\mathbf{E}(X|Y = i)$ for each $i \in \{1, 2, 3\}$, in terms of $\mathbf{E}X$.
- Compute $\mathbf{E}X$.