Probability, 170A, Spring 2017, UCLA   Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 15 pages (including this cover page) and 11 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	14	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total:	114	

[a]June 14, 2017,

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

   (a) (3 points) Let X be a continuous random variable with PDF f_X. Then, for any $x \in \mathbf{R}$, $\frac{d}{dx}\mathbf{P}(X \le x)$ exists, and

$$\frac{d}{dx}\mathbf{P}(X \le x) = f_X(x)$$

TRUE FALSE (circle one)

   (b) (3 points) Let X be a random variable such that

$$\mathbf{P}(X \le x) = \begin{cases} 0 & \text{, if } x < 0 \\ x^3 & \text{, if } 0 \le x \le 1. \\ 1 & \text{, if } x \ge 1 \end{cases}$$

Then $\mathbf{E}X = \frac{1}{15}$.

TRUE FALSE (circle one)

(c) (4 points) Let X, Y and Z be random variables. Suppose these random variables have joint density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} y+z & \text{, if } 0 \leq x,y,z \leq 1, \\ 0 & \text{, otherwise.} \end{cases}$$

Then $\mathbf{E}X = \frac{1}{2}$.

TRUE FALSE (circle one)

(d) (4 points) Let X and Y be random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume that X and Y are independent (with respect to the probability law $\mathbf{P}$). Let $\mathbf{P}'$ be another (possibly different) probability law on Ω. Then X and Y are independent, with respect to $\mathbf{P}'$.

(We say X and Y are independent with respect to $\mathbf{P}$ if we use $\mathbf{P}$ in the definition of independence of X and Y.)

TRUE FALSE (circle one)

2. (10 points) Two people are flipping fair coins. Let n be a positive integer. Person I flips $n+1$ coins. Person II flips n coins. Show that the following event has probability $1/2$: Person I has more heads than Person II.

3. (10 points) Suppose you drive a car with 100 tires. Suppose all of the tires are removed. Then, the mechanic now puts the tires back on the car randomly, so that all arrangements of the tires are equally likely. With what probability will no tire end up in its original position?

4. (10 points) Let X_1 be a geometric random variable with parameter p_1. (So, if k is a positive integer, then $\mathbf{P}(X_1 = k) = (1 - p_1)^{k-1} p_1$.) Let X_2 be a geometric random variable with parameter p_2. Assume that X_1 and X_2 are independent.

   - Compute $\mathbf{P}(X_1 \geq X_2)$.
   - Compute $\mathbf{P}(X_1 = X_2)$

   (You should simplify your answers as best you can.)

5. (10 points) You have three boxes. The first one contains 1 white and 8 black balls, the second one contains 5 white and 4 black balls, and the last one contains 2 white and 1 black ball. You choose one of these three boxes uniformly at random, and then pick a ball from this box also uniformly at random. What is the probability you pick a white ball?

6. (10 points) Let $a < b$ be fixed real numbers. Let X be a random variable which is uniformly distributed in the interval $[a, b]$. Compute the mean and variance of X. (As usual, you must show your work to receive credit.) (It is recommended, though not necessary, to simplify your answer as much as possible.)

7. (10 points) Let $c \in \mathbf{R}$. Let X, Y be continuous random variables with joint density $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \begin{cases} c & \text{, if } 0 \leq y < x < 1 \\ c & \text{, if } 0 \leq x \leq 1 \text{ and } x \leq y < 1-x \\ 0 & \text{, otherwise.} \end{cases}$$

Find the following three things:

- c
- f_X
- $f_{Y|X}$

8. (10 points) Let X be a positive discrete random variable. Prove:

$$\mathbf{E}\log(X) \leq \log(\mathbf{E}X).$$

9. (10 points) Let X, Y, Z be independent continuous random variables such that

- $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\forall\ x \in \mathbf{R}$.
- Y is uniformly distributed in $[-2, 2]$.
- $f_Z(z) = \frac{1}{c} e^{-z^{10}}$, $\forall\ z \in \mathbf{R}$. (Here c is a constant chosen so that $\int_{\mathbf{R}} f_Z(z) dz = 1$.)

Compute $\mathbf{P}(X + Y + Z < 0)$.

10. (10 points) Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let $Y = \max(X_1, \ldots, X_n)$ be the maximum of $X_1, \ldots, X_n$. Write an integral expression that computes $\mathbf{E}Y$. You should **not** try to evaluate this integral. This integral should be an expression involving the density $e^{-x^2/2}/\sqrt{2\pi}$. (Hint: in order to do this problem, you need to compute a derivative.)

11. (10 points) Suppose you have a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$, $a_1 = 1$, and such that, for any $n \geq 2$, we have

$$a_n = 2a_{n-1} + 3a_{n-2}.$$

Using the linear algebraic technique for solving recursions discussed for the Gambler's Ruin problem, find an explicit expression for a_n for any $n \geq 2$. (Hint: If you set things up exactly as we did in class, then the eigenvalues you get should be integers. You should also be able to choose eigenvectors with integer entries.) (You do not necessarily have to use the exact technique we used in class, but you need to fully justify whatever you do.)

(Scratch paper)

(More scratch paper)