Probability, 170A, Spring 2016, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ____________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 17 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	30	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total:	140	

[a] June 9, 2016,

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

   (a) (2 points) There exists a continuous random variable X such that $\mathbf{P}(X = 2) = 1/2$.

   TRUE FALSE (circle one)

   (b) (2 points) There is some random variable X such that $\text{var}(X) = -1$.

   TRUE FALSE (circle one)

   (c) (2 points) Let X, Y be continuous random variables. Then

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} \mathbf{E}(X|Y = y) f_X(y) dy.$$

   TRUE FALSE (circle one)

   (d) (2 points) Let X be a continuous random variable with probability density function f_X. Then $f_X(x) \leq 1$ for all $x \in \mathbf{R}$.

   TRUE FALSE (circle one)

   (e) (2 points) Define $f(x, y) = \begin{cases} 6e^{-3x-2y} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Suppose X and Y are random variables with joint PDF $f(x, y)$. Then X and Y are independent.

   TRUE FALSE (circle one)

(f) (3 points) Let X be a random variable uniformly distributed on the interval $[0, 1]$. Let $Y = -\log X$. (Here log denotes the natural logarithm.) Then Y has CDF given by

$$F_Y(y) = \mathbf{P}(Y \leq y) = \begin{cases} 0 & , y < 0 \\ 1 - e^{-y} & , 0 \leq y. \end{cases}$$

TRUE FALSE (circle one)

(g) (3 points) Let X, Y and Z be random variables. Suppose these random variables have joint density function

$$f_{X,Y,Z}(x, y, z) = \begin{cases} \frac{1}{6}(xy + z) & , \text{if } 0 \leq x, y, z \leq 2, \\ 0 & , \text{otherwise.} \end{cases}$$

Then $\mathbf{P}(X \leq 1, Y \leq 1, Z \leq 1) = 1/8$.

TRUE FALSE (circle one)

(h) (3 points) Let X be a random variable that only takes nonnegative integer values. Assume that for any integer $n > 10$, we have $\mathbf{P}(X \geq n) = 1/\sqrt{n}$. Then $\mathbf{E}(X) < \infty$.

TRUE FALSE (circle one)

(i) (3 points) For any $x \in \mathbf{R}$, define $F(x) = \frac{\pi}{2} + \tan^{-1}(x)$. Then there exists a random variable X such that $\mathbf{P}(X \leq x) = F(x)$ for all $x \in \mathbf{R}$. (Here $\tan^{-1}$ denotes the inverse tangent function.)

TRUE FALSE (circle one)

(j) (4 points) Let X and Y be random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume that X and Y are independent (with respect to the probability law $\mathbf{P}$). Let $\mathbf{P}'$ be another (possibly different) probability law on Ω. Then X and Y are independent, with respect to $\mathbf{P}'$.

(We say X and Y are independent with respect to $\mathbf{P}$ if we use $\mathbf{P}$ in the definition of independence of X and Y.)

TRUE   FALSE   (circle one)

(k) (4 points) Let X, Y be random variables. For any $y \in \mathbf{R}$, assume that

$$\mathbf{E}(X|Y = y) = |y| .$$

Also, assume that Y is a standard Gaussian random variable. Then $\mathbf{E}(X) = 2$.

TRUE   FALSE   (circle one)

2. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$. (So, X has density $f_X(x) = e^{-x}$ if $x \geq 0$, and $f_X(x) = 0$ if $x < 0$.) Compute $\mathbf{E}X$ and $\mathbf{E}(X^2)$.

3. (10 points) Let n be a fixed positive integer. Let X and Y be independent random variables that are uniformly distributed in the set $\{1, \ldots, n\}$. What is the PMF of $X + Y$?

To show your answer is correct, **write the following values on the next two lines**:
$p_{X+Y}(1) =$, $p_{X+Y}(1 + (n/3)) =$, $p_{X+Y}(n + 1) =$
$p_{X+Y}(1 + (3n/2)) =$, $p_{X+Y}(3n) =$.

4. (10 points) Let X, Y, Z be independent discrete random variables. Prove that X and Y are independent.

5. (10 points) A single fair 100-sided die has each of its faces labeled with exactly one integer between and including 1 and 100. Each face is equally like to be rolled.

   Suppose you have three fair 100-sided dice. Suppose you roll these three dice. What is the probability that the sum of the rolls of the three dice is 52?

6. (10 points) Let n be a fixed positive integer. Let $X_1, \ldots, X_n$ be independent random variables. As usual, define $\text{var}(X) = \mathbf{E}(X - \mathbf{E}X)^2$. Prove the following:

$$\text{var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{var}(X_i).$$

(As usual, you must show details to receive credit.)

7. (10 points) Let X be binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

   Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 4$ and $\mathbf{E}Y^2 = 32$.

   Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

   Let W be a discrete random variable such that $\mathbf{P}(W = -1) = 2/3$ and $\mathbf{P}(W = 2) = 1/3$, so that $\mathbf{E}W = 0$ and $\mathbf{E}W^2 = 2$.

   Assume that X, Y, Z and W are all independent. Compute

$$\mathbf{E}(1 + W^2 + WX^2Y^3Z^4).$$

8. (10 points) Let X, Y, Z be uniformly distributed random variables on $[0, 1]$. Assume that X, Y and Z are all independent. Compute the probability

$$\mathbf{P}(X + Y + Z < 2).$$

9. (10 points) Suppose I have a fair coin. So, each coin flip has probability 1/2 of landing heads, and probability 1/2 of landing tails. Suppose I flip the coin ten times, and each time it lands heads. When I flip the coin again an eleventh time, what is the probability that the coin lands heads?

10. (10 points) Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let $Y = \max(X_1, \ldots, X_n)$ be the maximum of $X_1, \ldots, X_n$. Write an integral expression that computes $\mathbf{E}Y$. You should **not** try to evaluate this integral. This integral should be an expression involving the density $e^{-x^2/2}/\sqrt{2\pi}$. (Hint: in order to do this problem, you need to compute a derivative.)

11. (10 points) Suppose you have a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$, $a_1 = 1$, and such that, for any $n \geq 2$, we have

$$a_n = 2a_{n-1} + 3a_{n-2}.$$

Using the linear algebraic technique for solving recursions discussed for the Gambler's Ruin problem, find an explicit expression for a_n for any $n \geq 2$. (Hint: If you set things up exactly as we did in class, then the eigenvalues you get should be integers. You should also be able to choose eigenvectors with integer entries.)

12. (10 points) Suppose you have a standard 52-card deck of playing cards. (So the cards are sitting in a deck one on top of the other; there are thirteen cards of each of the four suits: hearts, spades, diamonds and clubs. And all permutations of the cards as a single deck of cards are equally likely.) Suppose you are drawing cards from the top of the deck without replacing them, and you put the cards in a pile. What is the expected number of cards you have to draw from the top of the deck before you find **two hearts**? (That is, what is the expected number of cards you have to draw out of the deck right before the pile goes from having one heart to having two hearts?)

(Scratch paper)

(More scratch paper)