Probability, 170A, Spring 2016, UCLA — Instructor: Steven Heilman

Name: ____________ UCLA ID: ____________ Date: ____________

Signature: ____________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

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1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) The number of permutations of the set $\{1, 2, \ldots, n\}$ is $n!$.

   TRUE FALSE (circle one)

   (b) (2 points) Let X be a nonnegative random variable on a sample space Ω. Assume that X only takes integer values. Then

   $$\mathbf{E}(X) = \sum_{n=0}^{\infty} \mathbf{P}(X \geq n).$$

   TRUE FALSE (circle one)

   (c) (2 points) Let X and Y be discrete random variables with joint PMF $p_{X,Y}$. Assume that X and Y take a finite number of values. Then

   $$\mathbf{E}(X + Y) = (\mathbf{E}X) + (\mathbf{E}Y).$$

   TRUE FALSE (circle one)

   (d) (2 points) Let X and Y be discrete random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Suppose X and Y have joint PMF $p_{X,Y}$. Then, for any $x, y \in \mathbf{R}$, we have

   $$p_{X,Y}(x, y) = \mathbf{P}(X = x, Y = y).$$

   TRUE FALSE (circle one)

   (e) (2 points) Let $p = 2$. Then there exists a random variable X such that, for any integer k such that $0 \leq k \leq 100$, we have

   $$\mathbf{P}(X = k) = \binom{100}{k} p^k (1 - p)^{100-k}.$$

   TRUE FALSE (circle one)

2. (10 points) Let X be a random variable such that, for any positive integer $1 \leq k \leq 500$, we have $p_X(k) = 1/500$. Compute the probabilities of the following events.

- $X > 100$.
- $X^4 < 20$.
- $\sin(\pi X/2) = 0$.

3. (10 points) Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs thirty times, so that thirty spheres have been placed in bins. What is the expected number of empty bins at the end? (All of the sphere placements are independent of each other.)

4. (10 points) Let X be a discrete random variable. Prove:

$$\mathbf{E}(e^X) \geq e^{\mathbf{E}X}$$

5. (10 points) Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landings heads, and probability $1 - p$ of landing tails, each time it is flipped. Also, suppose you have a fair five-sided die (so each face of the cube has a distinct label from the set $\{1, 2, 3, 4, 5\}$, and each time you roll the die, any face of the die is rolled with equal probability.) Similarly, suppose you have a fair seven-sided die and a fair nine-sided die.

   Let N be the number of coin flips you need to do until the first head appears. If $N = 2$, roll the fair nine-sided die N times. If N is odd, roll the fair seven-sided die N times. If N is even and $N > 2$, then roll the fair five-sided die N times.

   Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$. (As usual, you must justify every step of your answer.) (Also, you can write your final answer using certain finite or infinite sums, without simplifying further.)

(Scratch paper)