170A Midterm 1 Solutions, Spring 2017[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) The negation of the statement "There exists an integer j such that $j^4 - j < 5$" is: "For every integer j, we have $j^4 - j \geq 5$."

TRUE, by the rules of negation, "There exists" is negated to "For every," and the inequality $<$ is negated to $\geq$.

(b) Let $\mathbf{P}$ be the uniform probability law on $[0,1]$. Let $x_1, x_2, \ldots \in [0,1]$ be a countable set of distinct points. Then

$$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = 0.$$

TRUE. By the definition of $\mathbf{P}$, $\mathbf{P}(\{x_n\}) = 0$ for all $n \geq 1$. So, from Axiom (ii) for probability laws,

$$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = \sum_{n=1}^{\infty}\mathbf{P}(\{x_n\}) = \sum_{n=1}^{\infty} 0 = 0.$$

(c) Let $\Omega = \{1,2,3,4,5,6,7\}$. For any $A \subseteq \Omega$, define $\mathbf{P}(A)$ to be the number of elements in A. Then $\mathbf{P}$ is a probability law on Ω.

FALSE. $\mathbf{P}(\Omega) = 7$, but Axiom (i) says $\mathbf{P}(\Omega) = 1$.

(d) Let $A_1, \ldots, A_n$ be disjoint events in a sample space Ω. That is, $A_i \cap A_j = \emptyset$ whenever $i, j \in \{1, \ldots, n\}$ satisfy $i \neq j$. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(A_i) > 0$ for all $1 \leq i \leq n$. Let $B \subseteq \Omega$. Then

$$\mathbf{P}(B) = \sum_{i=1}^{n}\mathbf{P}(B \cap A_i) = \sum_{i=1}^{n}\mathbf{P}(A_i)\mathbf{P}(B|A_i).$$

FALSE. Let $\Omega = \{1,2,3\}$, let $\mathbf{P}$ be uniform on Ω, let $A_1 = \{1\}$ and let $A_2 = \{2\}$. By Axiom (iii), $\mathbf{P}(\Omega) = 1$. $\sum_{i=1}^{2}\mathbf{P}(\Omega \cap A_i) = \sum_{i=1}^{2}\mathbf{P}(\{i\}) = 2/3$. So, $\mathbf{P}(\Omega) \neq \sum_{i=1}^{2}\mathbf{P}(B \cap A_i)$. The issue is, we also need to assume that $\cup_{i=1}^{n}A_i = \Omega$.

2. Question 2

Let $\Omega = [0,1]$. For any $A \subseteq \Omega$, define $\mathbf{P}(A)$ so that

$$\mathbf{P}(A) := \begin{cases} 1 & \text{, if } \frac{1}{3} \in A \\ 0 & \text{, if } \frac{1}{3} \notin A. \end{cases}$$

Verify that $\mathbf{P}$ is a probability law on Ω. (When you verify Axiom (ii), you should consider countable unions of sets.)

Solution. By the definition of $\mathbf{P}$, $\mathbf{P}(A) \geq 0$ for any $A \subseteq \Omega$. So, Axiom (i) holds. Also, by definition of $\mathbf{P}$, since $1/3 \in \Omega$ we have $\mathbf{P}(\Omega) = 1$. So, Axiom (iii) holds. To verify Axiom (ii), let $A_1, A_2, \ldots \subseteq \Omega$ be disjoint events. We split into two cases.

Case 1. If $1/3 \notin \cup_{n=1}^{\infty} A_n$, then $1/3 \notin A_n$ for every $n \geq 1$, by definition of the union. By the definition of $\mathbf{P}$, we then have $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = 0$ and $\mathbf{P}(A_n) = 0$ for all $n \geq 1$. Therefore,

$$\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbf{P}(A_n).$$

Case 2. If $1/3 \in \cup_{n=1}^{\infty} A_n$, then $1/3 \in A_i$ for some $i \geq 1$, by definition of the union. Since the events $A_1, A_2, \ldots$ are disjoint, we have $1/3 \notin A_n$ for every $n \geq 1$ with $n \neq i$. By the definition of $\mathbf{P}$, we then have $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = 1$ and $\mathbf{P}(A_n) = 0$ for all $n \geq 1$ with $n \geq i$, and $\mathbf{P}(A_i) = 1$. Therefore,

$$\mathbf{P}(\cup_{n=1}^{\infty} A_n) = 1 = \mathbf{P}(A_i) = \mathbf{P}(A_i) + 0 + 0 + \cdots = \sum_{n=1}^{\infty} \mathbf{P}(A_n).$$

In any case, Axiom (ii) holds.

3. Question 3

Suppose a test for a disease is 97% accurate. That is, if you have the disease, the test will be positive with 97% probability. And if you do not have the disease, the test will be negative with 97% probability. Suppose also the disease is fairly rare, so that roughly 1 in $1,000$ people have the disease. If you test positive for the disease, with what probability do you actually have the disease? (Hint: let B be the event that you test positive for the disease. Let A be the event that you actually have the disease. Compute a conditional probability.)

Solution. Let B be the event that the test is positive. Let A be the event that you actually have the disease. We want to compute $\mathbf{P}(A|B)$. We have

$$\mathbf{P}(A|B) = \mathbf{P}(A \cap B)/\mathbf{P}(B) = (\mathbf{P}(A)/\mathbf{P}(B))\mathbf{P}(A \cap B)/\mathbf{P}(A) = (\mathbf{P}(A)/\mathbf{P}(B))\mathbf{P}(B|A).$$

We are given that $\mathbf{P}(A) = 10^{-3}$, $P(B|A) = .97$ and $\mathbf{P}(B|A^c) = .03$. To compute $\mathbf{P}(B)$, we write $B = (B \cap A) \cup (B \cap A^c)$, so that

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(B \cap A) + \mathbf{P}(B \cap A^c) = \mathbf{P}(B|A)\mathbf{P}(A) + \mathbf{P}(B|A^c)\mathbf{P}(A^c)\\ &= .97(10^{-3}) + .03(1 - \mathbf{P}(A)) = .97(10^{-3}) + .03(1 - 10^{-3}) \approx 3 \times 10^{-2}.\end{aligned}$$

In conclusion,

$$\mathbf{P}(A|B) = \frac{10^{-3}}{\mathbf{P}(B)}(.97) = \frac{10^{-3}}{.97(10^{-3}) + .03(1 - 10^{-3})}(.97) \approx 10^{-3}10^{2}(1/3) = \frac{1}{30}.$$

4. Question 4

An urn contains three red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional red cube is put into the urn, and then a second cube is removed from the urn.

- What is the probability that the second cube removed from the urn is red?
- If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

Solution. Let A be the event that the first cube removed is red, and let B be the event that the first cube removed is blue. Let C be the event that the second cube removed from the urn is red. Then $A \cap B = \emptyset$ and $A \cup B = \Omega$, so the Total Probability Theorem says

$$\mathbf{P}(C) = \mathbf{P}(C|A)\mathbf{P}(A) + \mathbf{P}(C|B)\mathbf{P}(B) = (1/2)(3/5) + (4/6)(2/5) = 3/10 + 8/30 = 17/30.$$

Now, using that $\mathbf{P}(C) = 17/30$, we have

$$\mathbf{P}(B|C) = \mathbf{P}(C|B)[\mathbf{P}(B)/\mathbf{P}(C)] = (4/6)(2/5)(30/17) = 8/17.$$

5. Question 5

Let A, B, C be independent events. Show that A, B and C^c are independent events.

Solution. It is given that $\mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C)$. So, using $\mathbf{P}(A) = \mathbf{P}(A \cap C) + \mathbf{P}(A \cap C^c)$,

$$\mathbf{P}(A \cap C^c) = \mathbf{P}(A) - \mathbf{P}(A \cap C) = \mathbf{P}(A) - \mathbf{P}(A)\mathbf{P}(C) = \mathbf{P}(A)(1 - \mathbf{P}(C)) = \mathbf{P}(A)\mathbf{P}(C^c).$$

In the last equality, we used $\mathbf{P}(C) + \mathbf{P}(C^c) = 1$.

Similarly, using the assumption $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$, we have

$$\mathbf{P}(B \cap C^c) = \mathbf{P}(B) - \mathbf{P}(B \cap C) = \mathbf{P}(B) - \mathbf{P}(B)\mathbf{P}(C) = \mathbf{P}(B)(1 - \mathbf{P}(C)) = \mathbf{P}(B)\mathbf{P}(C^c).$$

To verify that A, B and C^c are independent events, it therefore remains to show that

$$\mathbf{P}(A \cap B \cap C^c) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C^c).$$

To show this, using the assumption $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$ and using also $\mathbf{P}(A \cap B) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B \cap C^c)$,

$$\begin{aligned}\mathbf{P}(A \cap B \cap C) &= \mathbf{P}(A \cap B \cap C) - \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B) \\ &= \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(1 - \mathbf{P}(C)) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C^c).\end{aligned}$$