170A Midterm 1 Solutions, Spring 2016[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) The negation of the statement "For every integer j, $j^2 + j \geq 0$" is:
"There exists an integer j such that $j^2 + j \leq 0$."

FALSE. Using the rules for negating statements, the negation is: There exists an integer j such that $j^2 + j < 0$.

(b) Let $A_1, A_2, \ldots$ be sets in a nonempty universe Ω. If $x \in \left(\cup_{j=1}^{\infty} A_j\right)^c$, then $x \in \cap_{j=1}^{\infty} A_j$.

FALSE. Let $\emptyset = A_1 = A_2 = \cdots$. Then $\cup_{j=1}^{\infty} A_j = \emptyset$, so $(\cup_{j=1}^{\infty} A_j)^c = \Omega$, whereas $\cap_{j=1}^{\infty} A_j = \emptyset$. So, any $x \in \Omega$ does not satisfy $x \in \emptyset$. So, the statement is false.

(c) Let A, B be subsets of a sample space Ω. Assume that $\mathbf{P}(B) > 0$. If $\mathbf{P}(A|B) = \mathbf{P}(B)$, then the sets A and B are independent.

FALSE. If $\Omega = \{1, 2, 3, 4\}$, if $\mathbf{P}$ is uniform, if $A = \{1\}$ and if $B = \{1, 2\}$, then $\mathbf{P}(A|B) = \mathbf{P}(A \cap B)/\mathbf{P}(B) = (1/4)/(1/2) = 1/2 = \mathbf{P}(B)$, but A, B are not independent, since $\mathbf{P}(A \cap B) = 1/4 \neq (1/4)(1/2) = \mathbf{P}(A)\mathbf{P}(B)$.

(d) Let $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. For any $A \subseteq \Omega$, define $\mathbf{P}(A)$ to be the number of elements in A. Then $\mathbf{P}$ is a probability law on Ω.

FALSE. $\mathbf{P}(\Omega) = 7 \neq 1$, so the third axiom for probability laws does not hold.

(e) $[0, 1] = \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$.

TRUE. If $0 \leq x \leq 1$, then $x \in \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$ for any positive integer j. So $x \in \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$, by the definition of countable intersection. That is, $[0, 1] \subseteq \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$. On the other hand, if $x \in \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$, then $-1/j \leq x \leq 1 + 1/j$ for every positive integer j. The only $x \in \mathbf{R}$ satisfying this condition are $x \in [0, 1]$. (By the Archimedean property of the real numbers, if $x < 0$ then there exists a positive integer j such that $x < -1/j < 0$; and if $x > 1$, then there exists a positive integer j such that $x > 1 + 1/j$.) So, $\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right) \subseteq [0, 1]$. In conclusion, $[0, 1] = \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$.

2. Question 2

Prove the following assertion by induction on n:

For any positive integer n, we have $1 + 2 + 3 + \cdots + n = n(n + 1)/2$.

Solution. We first check the base case $n = 1$. In this case, we have $n(n+1)/2 = 1(2)/2 = 1$. So, the base case holds. We now check the inductive step. Suppose $1 + 2 + 3 + \cdots + n =$

[1]April 23, 2016,

$n(n+1)/2$ for some positive integer n. We now consider the case $n+1$. Then

$$\begin{aligned} 1+2+\cdots+n+(n+1) &= [1+2+\cdots+n]+(n+1) \\ &= \frac{n(n+1)}{2}+(n+1) \qquad \text{, by the inductive hypothesis} \\ &= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

That is, the desired assertion holds in the case $n+1$. Since the inductive step has concluded, the assertion holds for all positive integers n.

3. Question 3

Suppose I have a bin with exactly 3 red, 4 green, 5 blue, and 6 yellow cubes. Step 1: I remove one cube from the bin uniformly at random, put the cube outside of the bin, then pause for one second. Step 2: I remove another cube from the bin uniformly at random, put the cube outside of the bin, then pause for one second. Step 3: I remove another cube from the bin uniformly at random, put the cube outside of the bin, then pause for one second. Let R be the event that a red cube is removed in Step 1. Let G be the event that a green cube is removed in Step 2. Let B be the event that a blue cube is removed in Step 3. Let A be the event $A = R\cap G\cap B$. What is $\mathbf{P}(A)$?

(As usual, you must justify your answer; also you do not need to simplify your final answer.)

Solution. We compute $\mathbf{P}(A)$ using the Multiplication rule.

$$\mathbf{P}(A) = \mathbf{P}(R\cap G\cap B) = \frac{\mathbf{P}(R\cap G\cap B)}{\mathbf{P}(R\cap G)}\frac{\mathbf{P}(R\cap G)}{\mathbf{P}(R)}\mathbf{P}(R) = \mathbf{P}(B|R\cap G)\mathbf{P}(G|R)\mathbf{P}(R).$$

Since the cube is removed uniformly at random in Step 1, we have $\mathbf{P}(R) = \frac{3}{3+4+5+6} = \frac{3}{18}$. Given that R has occurred, there are 2 red, 4 green, 5 blue and 6 yellow cubes in the container. Since the cube is removed uniformly at random in Step 2, we then have $\mathbf{P}(G|R) = \frac{4}{2+4+5+6} = \frac{4}{17}$. Given that R and G have occurred, there are 2 red, 3 green, 5 blue and 6 yellow cubes in the container. Since the cube is removed uniformly at random in Step 3, we then have $\mathbf{P}(B|R\cap G) = \frac{5}{2+3+5+6} = \frac{5}{16}$. In summary,

$$\mathbf{P}(A) = \frac{5}{16}\cdot\frac{4}{17}\cdot\frac{3}{18} = \frac{60}{4896} = \frac{5}{408}.$$

4. Question 4

Suppose a test for a disease is 98% accurate. That is, if you have the disease, the test will be positive with 98% probability. And if you do not have the disease, the test will be negative with 98% probability. Suppose also the disease is fairly rare, so that roughly 1 in $100,000$ people have the disease. If you test positive for the disease, with what probability do you actually have the disease? (Hint: Let B be the event that you test positive for the disease. Let A be the event that you actually have the disease. Compute a conditional probability.)

Solution. We want to compute $\mathbf{P}(A|B)$. From Bayes' Rule,

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A)}{\mathbf{P}(B)}\mathbf{P}(B|A) = \frac{\mathbf{P}(A)\mathbf{P}(B|A)}{\mathbf{P}(B|A)\mathbf{P}(A)+\mathbf{P}(B|A^c)\mathbf{P}(A^c)}.$$

It is given that $\mathbf{P}(A) = 10^{-5}$, $\mathbf{P}(B|A) = .98$, $\mathbf{P}(B|A^c) = .02$. Since $\mathbf{P}(A^c) + \mathbf{P}(A) = 1$, we have $\mathbf{P}(A^c) = 1 - 10^{-5}$. So,

$$\mathbf{P}(A|B) = \frac{10^{-5}(.98)}{.98(10^{-5}) + .02(1 - 10^{-5})} \approx \frac{10^{-5}}{.02} \approx \frac{1}{2} \cdot 10^{-3}.$$

5. Question 5

Two people are flipping fair coins. Let n be a positive integer. Person I flips $n+1$ coins. Person II flips n coins. Show that the following event has probability 1/2: Person I has more heads than Person II.

Solution 1. Let A be the event that Person I has more heads than Person II. Let S_I be the number of heads from the first n coin flips of person I. Let S_{II} be the number of heads from the first n coin flips of person II. Let B_1 be the event that the $(n+1)^{st}$ coin flip of person I is heads. Let B_2 be the event that the $(n+1)^{st}$ coin flip of person I is tails. Then $B_1 \cap B_2 = \emptyset$ since the $(n+1)^{st}$ coin flip of cannot be both heads and tails. And $B_1 \cup B_2 = \Omega$, since the $(n+1)^{st}$ coin flip must be either heads or tails. So, by the total probability theorem,

$$\mathbf{P}(A) = \mathbf{P}(A|B_1)\mathbf{P}(B_1) + \mathbf{P}(A|B_2)\mathbf{P}(B_2).$$

Now, since the $(n+1)^{st}$ coin flip is a fair coin, $\mathbf{P}(B_1) = \mathbf{P}(B_2) = 1/2$. That is,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(A|B_1) + \mathbf{P}(A|B_2)\right).$$

Given that B_1 occurs, the event A is equal to the event that $S_I \geq S_{II}$. Given that B_2 occurs, the event A is equal to the event $S_I > S_{II}$. So,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I > S_{II})\right).$$

Now, $\mathbf{P}(S_I > S_{II}) = \mathbf{P}(S_1 < S_{II})$ by symmetry (with respect to interchanging the roles of person I and person II). So,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I < S_{II})\right) = \frac{1}{2}.$$

In the last line, we used that the events $S_I \geq S_{II}$ and $S_I < S_{II}$ are disjoint, and their union is all of Ω, so $\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I < S_{II}) = 1$.

Solution 2. Let A be the event that Person I has more heads than Person II. Let B be the event that person I has more heads than person II after they both flip n coins. Let C be the event that person I has less heads than person II after they both flip n coins. Let D be the event that person I has the same number of heads as person II after they both flip n coins. Then $B \cap C = C \cap D = B \cap D = \emptyset$, since any such intersection involves mutually exclusive events. Also, $B \cup C \cup D = \Omega$, since after the players each flip n coins, one of the three events B, C, D must occur.

So, by the total probability theorem,

$$\mathbf{P}(A) = \mathbf{P}(A|B)\mathbf{P}(B) + \mathbf{P}(A|C)\mathbf{P}(C) + \mathbf{P}(A|D)\mathbf{P}(D).$$

Given that B has occurred, we already know that A has occurred, so that $\mathbf{P}(A|B) = 1$. Given that C has occurred, it is impossible for A to occur, so that $\mathbf{P}(A|C) = 0$. And given that D has occurred, person I has only one more coin flip; if it is a heads, then A occurs, and if it is tails, then A does not occur. Since the coin is fair, we conclude that $\mathbf{P}(A|C) = 1/2$. That is,

$$\mathbf{P}(A) = \mathbf{P}(B) + \frac{1}{2}\mathbf{P}(C) = \frac{1}{2}(2\mathbf{P}(B) + \mathbf{P}(C)).$$

To conclude, it remains to show that $2\mathbf{P}(B)+\mathbf{P}(C) = 1$. As noted already, $B\cap C = C\cap D = B \cap D = \emptyset$, and $B \cup C \cup D = \Omega$, so Axiom (ii) for Probability Laws says that

$$\mathbf{P}(B) + \mathbf{P}(C) + \mathbf{P}(D) = \mathbf{P}(B \cup C \cup D) = \mathbf{P}(\Omega) = 1.$$

Now, events B and D are symmetric with respect to relabeling the players I and II. Consequently, $\mathbf{P}(B) = \mathbf{P}(D)$. That is, $2\mathbf{P}(B) + \mathbf{P}(C) = 1$, as desired.

Solution 3. Let C_1 be the number of heads of Person I. Let C_2 be the number of heads of Person II. Let $A = \{C_1 > C_2\}$. Since $A\cup A^c = \Omega$ and $A\cap A^c = \emptyset$, we have $\mathbf{P}(A)+\mathbf{P}(A^c) = 1$. Note that $A^c = \{C_1 \leq C_2\}$. Since the coins are fair, the probability $\mathbf{P}(A^c)$ can be equivalently stated by relabeling the head and tail of the coin. That is, $\mathbf{P}(A^c)$ is equal to the probability of the event that Person I has less than or equal to the number of tails of Person II. The latter event is equal to $\{C_1 > C_2\}$. That is, $\mathbf{P}(A^c) = \mathbf{P}(C_1 > C_2) = \mathbf{P}(A)$. So, $2\mathbf{P}(A) = 1$, and $\mathbf{P}(A) = 1/2$.