Please provide complete and well-written solutions to the following exercises.

Due April 12th, in the discussion section.

Homework 2

Exercise 1. Show that in the game of chess, exactly one of the following situations is true:

- White has a winning strategy.
- Black has a winning strategy.
- Each of the two players has a strategy guaranteeing at least a draw.

You may assume that chess is progressively bounded. (Hint: you should not really need to use anything special about chess, other than that it is a partisan combinatorial game that is progressively bounded. Also, as usual, it is probably beneficial to start from a terminal position, and then work backwards, using induction.)

Exercise 2. We first describe the game of Y. In this game, there is an arrangement of white hexagons in an equilateral triangle. One player is assigned the color blue, and the other player is assigned the color yellow. The players then take turns filling in one hexagon at a time of their assigned color. The goal is to create a Y-shape that connects all three sides of the triangle. That is, the goal of the game is to have an unbroken path of a single color of hexagons that touches all three sides of the triangle.

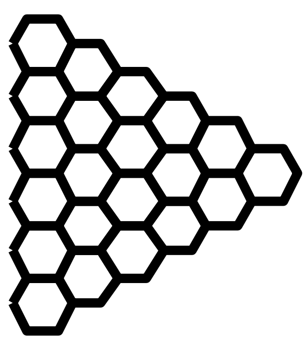

FIGURE 1. A Starting Position in the game of Y

Prove that the game of Hex can be realized as a special case of the game of Y. That is, the opening position on a standard hex board is equivalent to a particular game position in the game of Y. (Recall that we defined a notion for two games being equivalent.)

This game can be played on Windows at http://dbwilson.com/hex.exe .

Exercise 3. Let X be a random variable. Suppose X takes the value 1 with probability $1/3$; X takes the value 2 with probability $1/2$; and X takes the value 7 with probability $1/6$. Compute the expected value of X.

Exercise 4. Let X, Y be random variables taking the values 1 and 2 only. Let $p_{11} = 1/8$, $p_{12} = 3/8$, $p_{21} = 3/8$, $p_{22} = 1/8$. Assume that the probability that $X = i$ and that $Y = j$ is equal to p_{ij}. Are X and Y independent? Justify your answer.

Exercise 5. Let X, Y be random variables. Let $p_1, \ldots, p_n$ be nonnegative numbers with $\sum_{i=1}^n p_i = 1$. Let $q_1, \ldots, q_n$ be nonnegative numbers with $\sum_{i=1}^n q_i = 1$. Suppose X has value x_i with probability p_i for each $1 \leq i \leq n$. Suppose Y has value y_i with probability q_i for each $1 \leq i \leq n$. Show that the expected value of $X + Y$ is equal to the expected value of X, plus the expected value of Y. Or, using the notation EX to denote the expected value of X, show that $E(X + Y) = (EX) + (EY)$. (Note: in this exercise, it is *not* assumed that X and Y are independent.)

Exercise 6. Describe the optimal strategies for both players in rock-paper-scissors. Prove that these strategies are optimal. This game is described by the following payoff matrix.

Player I \ Player II	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Exercise 7. Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix

Player I \ Player II	A	B
C	0	2
D	4	1

Prove that these strategies are optimal.