167 Midterm 1 Solutions[1]

1. Question 1

(i) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: the first player has a winning strategy. (You only need to mention the game; you do not need to prove anything.)

Solution. Hex, Connect Four, and Chomp on a finite rectangle of size larger than 1×1 are all valid answers.

(ii) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: the second player has a winning strategy. (You only need to mention the game; you do not need to prove anything.)

Solution. Chomp played on a $2 \times \infty$ board, or on an $n \times \infty$ board for $n > 2$.

(iii) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: both players have a strategy guaranteeing at least a draw. (You only need to mention the game; you do not need to prove anything.)

Solution. Tic-tac-toe and Checkers are valid answers.

2. Question 2

Consider the game of Nim, where the game starts with four piles of chips. These piles have $1, 5, 3$ and 15 chips, respectively. Which player has a winning strategy from this position, the first player, or the second? Describe a winning first move.

Solution. In binary, the piles have 0001, 0011, 0101 and 1111 chips. So, the nim sum of the game position is $1000 \neq 0$. From Bouton's Theorem (Theorem 2.26 in the notes), the first player therefore has a winning strategy, and a winning first move is to force the nim sum to be zero. Such a move can be achieved by removing 8 chips from the pile that has 15 chips. The resulting game position will be $(1, 3, 5, 7)$. Or, in binary, the piles have 0001, 0011, 0101 and 0111 chips. So, the nim sum of this game position is zero.

3. Question 3

Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix. That is, at t optimal strategy, with what probability does player I play C, with what probability does player I play D, with what probability does player II play A, with what probability does player II play B?

Player I \ Player II	A	B
C	0	1
D	2	1

Prove that these strategies are optimal.

Solution. Let P denote the payoff matrix. By the Minimax Theorem and by definition of optimal strategy, the optimal strategies are vectors achieving $\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^T P y = \min_{y\in\Delta_2}\max_{x\in\Delta_2} x^T P y$. For example, the function $x \mapsto \min_{y\in\Delta_2} x^T P y$ achieves its maximum at an optimal strategy vector $x \in \Delta_2$. Write $x = (a, 1-a)$ and $y = (b, 1-b)$ where $a, b \in [0, 1]$.

Then $x^TPy = (a,1-a)^T(1-b,2b+(1-b)) = (a,1-a)^T(1-b,b+1) = a(1-b)+(1-a)(b+1) = 1+b-2ab$. So,

$$\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^TPy = \max_{a\in[0,1]}\min_{b\in[0,1]}(1+b-2ab).$$

Let $f(a,b) = 1+b-2ab$. Then $\nabla f(a,b) = (-2b, 1-2a)$. So, if $a \leq 1/2$, $\frac{\partial f}{\partial b} \geq 0$, and if $a \geq 1/2$, $\frac{\partial f}{\partial b} \leq 0$. That is, if $a \leq 1/2$, we have $\min_{b\in[0,1]}(1+b-2ab) = 1+(0)-2a(0) = 1$. And if $a \geq 1/2$, we have $\min_{b\in[0,1]}(1+b-2ab) = 1+(1)-2a(1) = 2-2a$. So, if

$$g(a) := \begin{cases} 1 & \text{if } a \leq 1/2 \\ 2-2a & \text{if } a \geq 1/2, \end{cases}$$

Then

$$\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^TPy = \max_{a\in[0,1]} g(a) = 1.$$

And this maximum is achieved for any $a \in [0,1/2]$. Also,

$$\min_{y\in\Delta_2}\max_{x\in\Delta_2} x^TPy = \min_{b\in[0,1]}\max_{a\in[0,1]}(1+b-2ab) = \min_{b\in[0,1]}(1+b) = 1.$$

So, the minimax occurs when $b = 0$ and when $a \in [0,1/2]$. That is, there are infinitely many optimal strategies, corresponding to $y = (0,1)$ and $x = (a,1-a)$, where $a \in [0,1/2]$. That is, player II will always play B, and for any $a \in [0,1/2]$, player I will play C with probability a and player I will play D with probability $1-a$.

4. Question 4

Let Y be a random variable such that: $Y = 2$ with probability $1/2$, $Y = 3$ with probability $1/2$.

Let Z be a random variable such that: $Z = 1$ with probability $1/2$ and $Z = 2$ with probability $1/2$. Assume that Z and Y are independent. What is the probability that: $Y = 3$ and $Z = 2$? What is the expected value of $Y \cdot Z$?

Solution. We know $Y = 3$ and $Z = 2$ with probability equal to: the probability $Y = 3$, multiplied by the probability $Z = 2$. So, the probability $Y = 3$ and $Z = 2$ is $(1/2)\cdot(1/2) = 1/4$, since Y and Z are independent, so these probabilities multiply.

Using similar reasoning, the expected value of $Y \cdot Z$ is $(1/4)(2\cdot 1)+(1/4)(2\cdot 2)+(1/4)(3\cdot 1)+(1/4)(3\cdot 2) = (1/4)(2+4+3+6) = 15/4$.

5. Question 5

Explicitly define some function $f\colon [0,1]\times[0,1] \to \mathbb{R}$ such that

$$\min_{y\in[0,1]}\max_{x\in[0,1]} f(x,y) \neq \max_{x\in[0,1]}\min_{y\in[0,1]} f(x,y).$$

Prove that the function f satisfies this property.

Solution. There are many possible ways to solve this problem. Here is one way. Let $(x,y) \in [0,1]^2$. Define

$$f(x,y) = \begin{cases} 1 & \text{if } x > 1/2 \text{ and } y > 1/2 \\ 1 & \text{if } x \leq 1/2 \text{ and } y \leq 1/2 \\ 0 & \text{if } x > 1/2 \text{ and } y \leq 1/2 \\ 0 & \text{if } x \leq 1/2 \text{ and } y > 1/2 \end{cases}$$

Note that the graph of f resembles a 2×2 checkerboard.

For any $y \in [0,1]$, there is an $x \in [0,1]$ such that $f(x,y) = 1$, so $\max_{x\in[0,1]} f(x,y) = 1$ for any $y \in [0,1]$, so $\min_{y\in[0,1]} \max_{x\in[0,1]} f(x,y) = 1$.

On the other hand, for any $x \in [0,1]$, there is an $y \in [0,1]$ such that $f(x,y) = 0$, so $\min_{y\in[0,1]} f(x,y) = 0$ for any $x \in [0,1]$, so $\max_{x\in[0,1]} \min_{y\in[0,1]} f(x,y) = 0$.