Please provide complete and well-written solutions to the following exercises.

Due November 22, in the discussion section.

Homework 7

Exercise 1. Let $E := \{(i,j) \in \{1,\ldots,n\} \colon i \neq j\}$. Show that the dual of the maximum flow problem is:

$$\text{minimize} \quad \sum_{(i,j)\in E} c_{ij}x_{ij} \quad \text{subject to the constraints}$$

$$\sum_{(i,j)\in p} x_{ij} \geq 1, \qquad \forall\, p \in P.$$

$$x_{ij} \geq 0, \qquad \forall\,(i,j) \in E.$$

Exercise 2 (Set Cover Problem)**.** Let U be a finite set, and let $S_1,\ldots,S_n$ be subsets of U such that $\cup_{i=1}^n S_i = U$. The goal of the set cover problem is to find the minimum number of sets $S_1,\ldots,S_n$ whose union is still all of U. For example, we could think of $S_1,\ldots,S_n$ as time intervals for shifts of workers, and we want to minimize the number of shifts that occur, while always having at least one person on the job.

For each $1 \leq i \leq n$, let $x_i = 1$ if we want to keep the set S_i, and $x_i = 0$ if we do not keep the set S_i. Then we can state the set cover problem as follows:

$$\text{minimize} \quad \sum_{i=1}^n x_i \quad \text{subject to the constraints}$$

$$\sum_{i\in\{1,\ldots,n\}\colon u\in S_i} x_i \geq 1, \quad \forall\, u \in U.$$

$$x_i \in \{0,1\}, \quad \forall\, i \in \{1,\ldots,n\}.$$

Show that the first constraint implies that $\cup_{i\in\{1,\ldots,n\}\colon x_i=1} S_i = U$.

We relax this integer program to a linear program:

$$\text{minimize} \quad \sum_{i=1}^n x_i \quad \text{subject to the constraints}$$

$$\sum_{i\in\{1,\ldots,n\}\colon u\in S_i} x_i \geq 1, \quad \forall\, u \in U.$$

$$0 \leq x_i \leq 1, \quad \forall\, i \in \{1,\ldots,n\}.$$

Find the dual of the linear program.

(Optional challenge: can you modify what we did for the minimum vertex cover problem, and create an approximation algorithm for the set cover problem? If so that would be nice, since the minimum vertex cover problem is also NP-complete.)

Exercise 3. Let K be the following set of positive semidefinite 2×2 matrices

$$K = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0 \colon a, b, c \in \mathbf{R},\ a + c = 1 \right\}.$$

First, show that K is convex and nonempty. Then show that K has infinitely many extreme points.

(Hint: which matrices in K have determinant zero?)

Exercise 4. Let $u, v \in \mathbf{R}^3$ be vectors such that $||u|| = ||v|| = 1$. Let $S^2 = \{x \in \mathbf{R}^3 \colon ||x|| = 1\}$. For any $t \in \mathbf{R}$, define

$$\mathrm{sign}(t) := \begin{cases} 1 & \text{, if } t > 0 \\ -1 & \text{, if } t \leq 0. \end{cases}$$

Let dS denote the surface area element on S^2. Show the following calculation, which was used in the analysis of the MAX-CUT semidefinite program.

$$\frac{\int_{S^2} \mathrm{sign}(\langle u, x\rangle) \cdot \mathrm{sign}(\langle v, x\rangle) dS(x)}{\int_{S^2} dS(x)} = 1 - \frac{2}{\pi} \cos^{-1}(\langle u, v\rangle).$$

Exercise 5 (Optional)**.** Using any numerical method you wish to use, compute

$$\min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta}$$

to ten decimal places of accuracy.