118 Final Solutions, Fall 2018[1]

1. Question 1

The table below summarizes some data about a real function $f(x)$.

It is also given information that $f''(x) > 0$ on the interval $(5, 7)$ and $f''(x) < 0$ on the interval $(7, 9)$, and $f''(x) > 0$ on the interval $(9, 10)$.

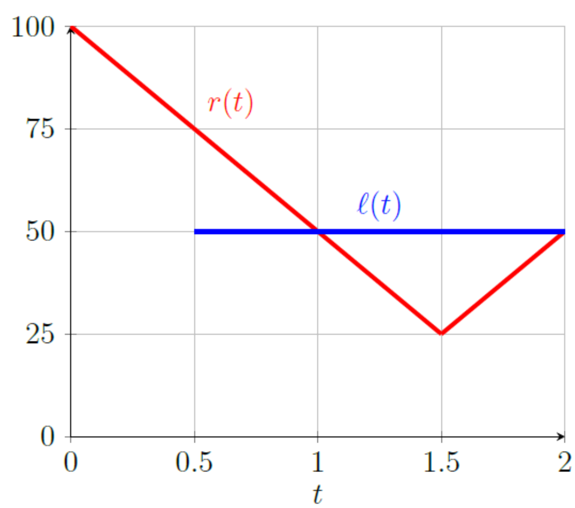
At each x-value from 1 to 10, determine whether or not $f(x)$ has a local maximum, local minimum, or inflection point there, and circle the appropriate response in each column. If it is not possible to make that determination using the data at hand, circle "Inconclusive."

You do not need to show any work for this question.

x	$f'(x)$	$f''(x)$	local max	local min	inflection point
1	0	-3	Yes	No	No
2	0	0	Inc	Inc	Inc
3	0	1	No	Yes	No
4	1	0	No	No	Inc
5	-1	2	No	No	No
6	0	2	No	Yes	No
7	4	0	No	No	Yes
8	1	-3	No	No	No
9	0	0	No	No	Yes
10	3	0	No	No	Inc

2. Question 2

A function $r(t)$ records the rate at which rain is falling into a bucket at time t, in cubic inches per hour. At $t = 0.5$ hours, the bucket springs a leak, and at each time t water is escaping at a rate of $\ell(t)$ cubic inches per hour. Shown below is a graph of both $r(t)$ and $\ell(t)$.



[1] December 19, 2018,

(a) If the bucket is empty at time $t = 0$, how much water is in the bucket at time $t = 2$? Justify your answer, either by obtaining it from an integral, or indicating how it comes from the graphs.

Solution. The amount of water is the area under the red curve $r(t)$ minus the area under the blue curve $\ell(t)$. This follows from the Fundamental Theorem of Calculus. If $R(t), L(t)$ are such that $R'(t) = r(t)$ and $L'(t) = \ell(t)$, then

$$R(2) - L(2) - [R(0) - L(0)] = \int_0^2 (R'(t) - L'(t))dt = \int_0^2 (r(t) - \ell(t))dt.$$

Since $R(0) = L(0) = 0$, we then have

$$R(2) - L(2) = \int_0^2 (r(t) - \ell(t))dt.$$

The last quantity is equal to

$$2(100)(1/2) + (1/2)(25) - (1.5)(50) = 100 + 12.5 - 75 = 37.5.$$

(b) When does the bucket contain the largest amount of water?

Solution. The bucket contains the most water at $t = 1$ hours. Before this time the water level in the bucket is rising, since the rate of in-flow is larger than the rate of out-flow. But after this time, the water level in the bucket is falling, since the rate of out-flow is larger than the rate of in-flow. That is $(d/dt)(R(t) - L(t)) > 0$ for $0 \leq t \leq 1$ and $(d/dt)(R(t) - L(t)) < 0$ for $1 < t < 2$. Therefore, the global maximum of $R(t) - L(t)$ occurs at $t = 1$.

(c) Construct an integral that gives the average rate at which rain enters the bucket between $t = 0$ and $t = 1.5$. You do not need to evaluate the integral.

$$\frac{\int_0^{1.5} r(t)dt}{\int_0^{1.5} dt} = \frac{\int_0^{1.5} r(t)dt}{1.5} = \frac{2(100)(1/2) + (1/2)(25)}{1.5} = \frac{225}{3}.$$

3. Question 3

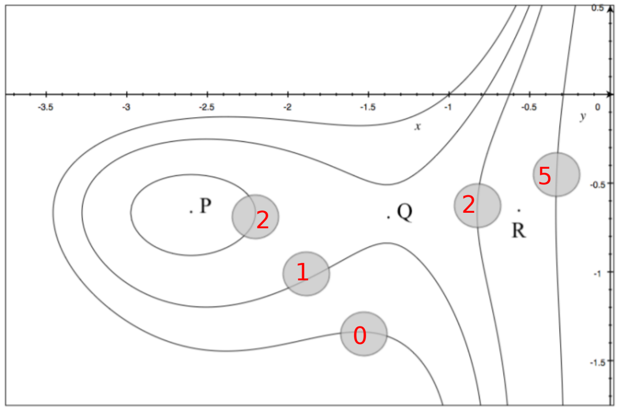
The contour diagram below shows the contours of a function $z = f(x, y)$ for the z-values $z = 0, 1, 2, 5$. There is a local maximum at P, a saddle point at Q, and these are the only critical points in this domain. If we also know that $f_x(R) > 0$, label the contours by putting the correct z-value in the grey circles that overlap the contour. Each z-value may be used more than once. You do not need to show any work for this question.

4. Question 4

A carpenter sells desks for \$30 each and, at this price, sells 100 desks per month. The carpenter estimates that for each \$5 increase in price, she sells 10 fewer desks per month. If the desks are manufactured at a cost of \$2 per desk, at what price should they be sold to generate the greatest possible profit?

Carefully show why your answer yields the greatest possible profit and show all of your work.

Solution. If the price is x, then the quantity $Q(x)$ sold is linear in x, as stated in the problem. So, $Q(x) = ax + b$ for some constant a, b. We know that $Q(x + 5) = Q(x) - 10$, so that $a(x + 5) + b = ax + b - 10$, so $5a = -10$ and $a = -2$. So, $Q(x) = -2x + b$. It is also given that $Q(30) = 100$, so that $-60 + b = 100$, i.e. $b = 160$. So, $Q(x) = -2x + 160$.



The profit $P(x)$ is then revenue minus cost, so

$$P(x) = [x-2]Q(x) = (x-2)(160-2x) = -2x^2+4x+160x-320 = -2x^2+164x-320.$$

Then $P'(x) = -4x+164$. So, $P'(x) = 0$ when $x = 41$, $P'(x) > 0$ when $x < 41$ and $P'(x) < 0$ when $x > 41$. By the first derivative test, $P(x)$ then achieves its maximum value at $x = 41$. This is the price the carpenter should sell the desks.

5. Question 5

The following table is included with every new California Drivers License. It shows the Blood Alcohol Content $B(d, w)$ for males and females as a function of the number of drinks they consume, d, and their body weight in pounds, w. It is illegal in California to drive with a BAC of 0.08% or higher. You rode to a party with a female friend who just had one 12oz beer and weighs 140 lbs.

BLOOD ALCOHOL CONTENT (BAC)
Table for Male (M) / Female (F)

Number of Drinks		Body Weight in Pounds								Driving Condition
		100	120	140	160	180	200	220	240	
0	M	.00	.00	.00	.00	.00	.00	.00	.00	Only Safe Driving Limit
	F	.00	.00	.00	.00	.00	.00	.00	.00	
1	M	.06	.05	.04	.04	.03	.03	.03	.02	Driving Skills Impaired
	F	.07	.06	.05	.04	.04	.03	.03	.03	
2	M	.12	.10	.09	.07	.07	.06	.05	.05	
	F	.13	.11	.09	.08	.07	.07	.06	.06	
3	M	.18	.15	.13	.11	.10	.09	.08	.07	
	F	.20	.17	.14	.12	.11	.10	.09	.08	Legally Intoxicated
4	M	.24	.20	.17	.15	.13	.12	.11	.10	
	F	.26	.22	.19	.17	.15	.13	.12	.11	
5	M	.30	.25	.21	.19	.17	.15	.14	.12	
	F	.33	.28	.24	.21	.18	.17	.15	.14	

Subtract .01% for each 40 minutes that lapse between drinks.
1 drink = 1.5 oz. 80 proof liquor, 12 oz. 5% beer, or 5 oz. 12% wine.
Fewer than 5 persons out of 100 will exceed these values.

(a) For females, use the data to estimate the value of $B_w(1, 140)$. As an estimate, different answers can receive full credit as long as method is correct.

Solution 1. We have the approximation

$$B_w(1, 140) \approx \frac{B(1, 160) - B(1, 140)}{160 - 140} = \frac{.04 - .05}{20} = -\frac{.01}{20}.$$

Solution 2. We have the approximation

$$B_w(1, 140) \approx \frac{B(1, 140) - B(1, 120)}{140 - 120} = \frac{.05 - .06}{20} = -\frac{.01}{20}.$$

Solution 3. We have the approximation

$$B_w(1,140) \approx \frac{B(1,160) - B(1,120)}{160-120} = \frac{.04 - .06}{20} = -\frac{.02}{40}.$$

(b) For females, use the data to estimate the value of $B_d(1,140)$. As an estimate, different answers can receive full credit as long as method is correct.

Solution 1. We have the approximation

$$B_d(1,140) \approx \frac{B(2,140) - B(1,140)}{2-1} = \frac{.09 - .05}{2-1} = .04.$$

Solution 2. We have the approximation

$$B_d(1,140) \approx \frac{B(1,140) - B(0,140)}{1-0} = \frac{.05 - .00}{2-1} = .05.$$

Solution 3. We have the approximation

$$B_d(1,140) \approx \frac{B(2,140) - B(0,140)}{1-0} = \frac{.09 - .00}{2-0} = .045.$$

(c) Use your estimates from Parts (a) and (b) to estimate your friends BAC if she had another 8oz of beer (1.75 drinks total), and weighed 147 pounds. Can she legally drive?

Solution 1. For (d,w) near $(1,140)$, We have the linear approximation

$$B(d,w) \approx B(1,140) + (d-1)B_d(1,140) + (w-140)B_w(1,140)$$

$$B(1.75,147) \approx B(1,140)+(.75)B_d(1,140)+(7)B_w(1,140) = .05+.75(.04)+7(-.01)/20 = .0765.$$

Solution 2. For (d,w) near $(1,140)$, We have the linear approximation

$$B(d,w) \approx B(1,140) + (d-1)B_d(1,140) + (w-140)B_w(1,140)$$

$$B(1.75,147) \approx B(1,140)+(.75)B_d(1,140)+(7)B_w(1,140) = .05+.75(.05)+7(-.01)/20 = .0840.$$

Solution 3. For (d,w) near $(1,140)$, We have the linear approximation

$$B(d,w) \approx B(1,140) + (d-1)B_d(1,140) + (w-140)B_w(1,140)$$

$$B(1.75,147) \approx B(1,140)+(.75)B_d(1,140)+(7)B_w(1,140) = .05+.75(.045)+7(-.02)/40 = .0803.$$

(d) If a male friend had 2.3 drinks and weighed 173 pounds, could they legally drive?

Solution 1. For (d,w) near $(2,160)$, We have the linear approximation

$$B(d,w) \approx B(2,160) + (d-2)B_d(2,160) + (w-160)B_w(2,160)$$

$$B(2.3,173) \approx B(2,160) + (.3)B_d(2,160) + (13)B_w(2,160) = .07 + .3(.04) + 13(0) = .0820.$$

6. Question 6

Compute the derivatives indicated in each part without using a calculator. Show every step, and highlight the derivative rules that you use.

(a) Let $f(t) = (2^t + 5t)^9$.

$$\begin{aligned}
\frac{df}{dt} &= 9(2^t + 5t)^8 \frac{d}{dt}(2^t + 5t) && \text{, Chain Rule} \\
&= 9(2^t + 5t)^8 (2^t \ln(2) + 5).
\end{aligned}$$

The last line used the Derivative rule for Exponentials $(b^t)' = b^t \ln b$ and Polynomials $(t^n)' = nt^{n-1}$.

(b) Let $R(p, q) = \frac{q}{p^2+q^2}$.

$$\begin{aligned}
R_q &= \frac{(p^2 + q^2)\frac{\partial}{\partial q}(q) - q\frac{\partial}{\partial q}(p^2 + q^2)}{(p^2 + q^2)^2} && \text{, Quotient Rule} \\
&= \frac{(p^2 + q^2) - q(2q)}{(p^2 + q^2)^2} && \text{, Derivative Rule for Polynomials} \\
&= \frac{(p^2 - q^2}{(p^2 + q^2)^2}.
\end{aligned}$$

(c) Let $f(x, y) = xln(xy) = x(\ln(x) + \ln(y)) = x \ln x + x \ln y$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial}{\partial y}(x \ln y) && \text{, since the first part only depends on } x \\
&= \frac{\partial}{\partial x}\frac{x}{y} && \text{, Derivative Rule for ln} \\
&= \frac{1}{y} && \text{, Derivative Rule for polynomials}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\frac{\partial}{\partial x}(x \ln x + x \ln y) \\
&= \frac{\partial}{\partial y}(x)\frac{\partial}{\partial x} + \ln x \frac{\partial}{\partial x}(x) + \ln x + \ln y && \text{, Product Rule} \\
&= \frac{\partial}{\partial y}(x(1/x) + \ln x + \ln y) && \text{, Derivative Rule for ln} \\
&= \frac{\partial}{\partial y}(\ln y) && \text{, since the other terms only depend on } x \\
&= \frac{1}{y} && \text{, Derivative Rule for ln.}
\end{aligned}$$

7. Question 7

Compute the following definite or indefinite integrals without using a calculator. Show every step, and highlight the integration techniques that you use.

(a) We integrate directly using $\int x^n dx = x^{n+1}/n+1+C$ for $n \geq -1$ and $\int (1/x)dx = \ln x + C$ to get

$$\begin{aligned}\int_1^2 \frac{3x^5 - 2x^4 + 7x^3 - 5x^2 + 1}{x} dx &= \int_1^2 3x^4 - 2x^3 + 7x^2 - 5x + \frac{1}{x} dx \\ &= [\frac{3}{5}x^5 - \frac{1}{2}x^4 + \frac{7}{3}x^3 - \frac{5}{2}x^2 + \ln(x)]_{x=1}^{x=2} \\ &= \frac{3}{5}2^5 - 8 + \frac{7}{3}(8) - 10 + \ln(2) - \frac{3}{5} + \frac{1}{2} - \frac{7}{3} + \frac{5}{2} - 0.\end{aligned}$$

(b) Using the substitution $u = \ln q$ so that $du = (1/q)dq$ and

$$\begin{aligned}\int \frac{(\ln q + 1)^2}{q} dq &= \int (u+1)^2 du, &&\text{, by the substitutiong } u = \ln q \\ &= \frac{1}{3}(u+1)^3, &&\text{, finding the antiderivative directly} \\ &= \frac{1}{3}(\ln(q)+1)^3 + C.\end{aligned}$$

(c) We begin by integrating by parts $\int udv = uv - \int vdu$ with $u = t^2 - 3t + 4$ and $dv = e^{-3t}$, so that $du = 2t - 3$ and $v = -(1/3)e^{-3t}$ so that

$$\begin{aligned}\int_0^2 (t^2 - 3t + 4)e^{-3t} dt &= [(t^2 - 3t + 4)(-1/3)(e^{-3t})]_{t=0}^{t=2} - \int_0^2 (-1/3)(2t-3)e^{-3t} dt \\ &= (-1/3)[4 - 6 + 4]e^{-6} + 4 + (1/3)\int_0^2 (2t-3)e^{-3t} dt \\ &= -(2/3)e^{-6} + 4 + (1/3)\int_0^2 (2t-3)e^{-3t} dt.\end{aligned}$$

We again integrate by parts $\int udv = uv - \int vdu$ with $u = 2t - 3$ and $dv = e^{-3t}$, so that $du = 2$ and $v = -(1/3)e^{-3t}$ so that

$$\begin{aligned}\int_0^2 (2t-3)e^{-3t} &= [(2t-3)(-1/3)(e^{-3t})]_{t=0}^{t=2} - \int_0^2 (-1/3)(2)e^{-3t} dt \\ &= (-1/3)e^{-6} - 1 + (2/3)\int_0^2 e^{-3t} dt = (-1/3)e^{-6} - 1 - (2/9)[e^{-3t}]_{t=0}^{t=2} \\ &= (-1/3)e^{-6} - 1 - (2/9)(e^{-6} - 1).\end{aligned}$$

In summary,

$$\int_0^2 (t^2 - 3t + 4)e^{-3t} dt = -(2/3)e^{-6} + 4 + (1/3)[(-1/3)e^{-6} - 1 - (2/9)(e^{-6} - 1)].$$

8. Question 8

Find all critical points of the following function. Classify each critical point using the second derivative test:

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Solution. We solve for $f_x = f_y = 0$. That is,

$$6xy - 6x = 0, \qquad 3x^2 + 3y^2 - 6y = 0.$$

From the first equation, $6xy = 6x$. So, if $x \neq 0$, we have $y = 1$. Then the second equation says that $3x^2 + 3 - 6 = 0$, so that $3x^2 = 3$ and $x = \pm 1$. In the remaining case that $x = 0$, the last equation says $3y^2 - 6y = 0$, so that $3y^2 = 6y$. So, $y = 0$ or $3y = 6$ so that $y = 1/2$. We have found four critical points:

$$(1,1), \qquad (-1,1), \qquad (0,0), \qquad (0,1/2).$$

We have

$$\begin{aligned} D(x,y) &= \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - f_{xy}^2 \\ &= (6y-6)(6y-6) - (6x)^2 = (6y-6)^2 - 36x^2. \end{aligned}$$

The critical point $(1,1)$ has $D(1,1) = -36 < 0$, so $(1,1)$ is a saddle point.

The critical point $(-1,1)$ has $D(-1,1) = -36 < 0$, so $(-1,1)$ is a saddle point.

The critical point $(0,0)$ has $D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$. So, $(0,0)$ is a local max.

The critical point $(0,1/2)$ has $D(0,1/2) = 9 > 0$ and $f_{xx}(0,0) = 3 - 6 < 0$. So, $(0,1/2)$ is a local max.

9. Question 9

Find the maximum/minimum values of the function $f(x,y) = 3xy$, when the variables x and y must also satisfy the constraint $2x^2 + y^2 = 1$. For each point you find that is a candidate for a max/min, identify it as a max, min or neither.

Solution. We solve the system of equations $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $g = 0$ where $g(x,y) = 2x^2 + y^2 - 1$. That is, we solve

$$\begin{cases} 3y = \lambda 4x \\ 3x = \lambda 2y \\ 2x^2 + y^2 = 1. \end{cases}$$

The first equation says $y = (4/3)\lambda x$. Substituting into the second equation gives

$$3x = (8/3)\lambda^2 x.$$

So, if $x \neq 0$, we have $\lambda^2 = (9/8)$. Substituting the first equation into the last one gives

$$2x^2 + (16/9)\lambda^2 x^2 = 1.$$

$$x^2(2 + (16/9)\lambda^2) = 1.$$

Since $\lambda^2 = 9/8$, we have

$$x^2(2+2) = 1.$$

That is, $x^2 = 1/4$. The last equation then says that $y^2 = 1 - 2x^2 = 1 - 1/2 = 1/2$. So, $y^2 = 1/2$. In summary, there are four candidate maxima/minima:

$$(1/2, 1/\sqrt{2}), \qquad (1/2, -1/\sqrt{2}), \qquad (-1/2, 1/\sqrt{2}), \qquad (-1/2, -1/\sqrt{2}).$$

We plug in these points into the function $f(x, y)$ to get

$$f(1/2, 1/\sqrt{2}) = 3/(2\sqrt{2}), \qquad f(1/2, -1/\sqrt{2}) = -3/(2\sqrt{2})$$
$$f(-1/2, 1/\sqrt{2}) = -3/(2\sqrt{2}), \qquad f(-1/2, -1/\sqrt{2}) = 3/(2\sqrt{2}).$$

So, the maximum occurs value of $3/(2\sqrt{2})$ occurs at $(1/2, 1/\sqrt{2})$ and $(-1/2, -1/\sqrt{2})$, and the minimum value of $-3/(2\sqrt{2})$occurs at $(-1/2, 1/\sqrt{2})$ and $(1/2, -1/\sqrt{2})$.

10. Question 10

Let $f(x, y) = 3x + 5y$, and let R be the region in the x, y-plane bounded by the curves

$$y = 4x^2,$$
$$y = 12x$$

Set up the integral $\iint_R f(x, y)dA$ in two ways, with the appropriate bounds:

(a) ... as an iterated integral in the order dx dy,

Solution. The curves intersect when $4x^2 = 12x$ i.e when $x = 0$ or when $4x = 12$, so that $x = 3$. When $0 \leq x \leq 3$, $12x \geq 4x^2$ (e.g. plug in $x = 1$ to see this). When $x = 0$, both take the y values 0. And when $x = 3$, both take the y value 36. If we first integrate in x, we begin at $x = y/12$ and we increase x until we hit the (positive) x value $x = \sqrt{y/4}$. So,

$$\iint_R f(x, y)dA = \int_{y=0}^{y=36} \int_{x=y/12}^{x=\sqrt{y/4}} (3x + 5y)dxdy.$$

(b) ... as an iterated integral in the order dy dx,

Solution. The curves intersect when $4x^2 = 12x$ i.e when $x = 0$ or when $4x = 12$, so that $x = 3$. When $0 \leq x \leq 3$, $12x \geq 4x^2$ (e.g. plug in $x = 1$ to see this). So, if we first integrate in y, we begin at $y = 4x^2$ and increase y until we hit the larger value of $12x$. So,

$$\iint_R f(x, y)dA = \int_{x=0}^{x=3} \int_{y=4x^2}^{y=12x} (3x + 5y)dydx$$

(c) Compute $\iint_R f(x, y)dA$, using the ordering that you prefer.

Solution 1.

$$\begin{aligned}
\iint_R f(x, y)dA &= \int_{x=0}^{x=3} \int_{y=4x^2}^{y=12x} (3x + 5y)dydx \\
&= \int_{x=0}^{x=3} [3xy + (5/2)y^2]_{y=4x^2}^{y=12x} dx \\
&= \int_{x=0}^{x=3} 36x^2 + (5/2)144x^2 - 12x^3 - (5/2)(16x^4)dx \\
&= [12x^3 + (5/6)144x^3 - 3x^4 - 8x^5]_{x=0}^{x=3} = 12(27) + (5/6)(27)(144) - 3^5 - 8(3^5) = 1377.
\end{aligned}$$

Solution 2.

$$\begin{aligned}
\iint_R f(x,y)dA &= \int_{y=0}^{y=36}\int_{x=y/12}^{x=\sqrt{y/4}}(3x+5y)dxdy \\
&= \int_{y=0}^{y=36}[(3/2)x^2+5xy]_{x=y/12}^{x=\sqrt{y/4}}dy \\
&= \int_{y=0}^{y=36} 3y/8+(5/2)y^{3/2}-(3/2)y^2/144-5y^2/12\ dy \\
&= [3y^2/16+y^{5/2}-(1/2)y^3/144-5y^3/36]_{y=0}^{y=36} \\
&= 3(1296)/16+6^5-(1/2)(36)^3/144-5(1296)=1377.
\end{aligned}$$

(d) Construct a formula for the average value of $f(x,y)$ over the region R. You do not need to evaluate this formula.

$$\begin{aligned}
\frac{\iint_R f(x,y)dA}{\iint_R dA} &= \frac{1377}{\int_{x=0}^{x=3}\int_{y=4x^2}^{y=12x} dydx} = \frac{1377}{\int_{x=0}^{x=3} 12x-4x^2\,dx} \\
&= \frac{1377}{[6x^2-(4/3)x^3]_{x=0}^{x=3}} = \frac{1377}{6(9)-4(9)} = \frac{1377}{18}
\end{aligned}$$